Definition: A smooth dynamical system on $\mathbb{R}^n$ is a continuous differentiable function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, where $\phi(t,x) = \phi_t(x)$ satisfies

1) $\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\phi_0(x) = x$
2) $\phi_t \circ \phi_s(x) = \phi_t(\phi_s(x)) = \phi_{t+s}(x)$

Ex: $x' = ax \qquad \phi_t(x_0) = x_0 e^{at}$

$\phi_t(x) = x e^{at}$

1) $\phi_0(x_0) = x_0 e^{a0} = x_0$ ✓
2) $\phi_t \circ \phi_s(x_0) = \phi_t(\phi_s(x_0)) = \phi_s(x_0) e^{at} =$

$= (x_0 e^{as}) e^{at} = x_0 e^{a(s+t)} =$

$= \phi_{t+s}(x_0)$ ✓

<u>Ex</u>: $\phi_t(x_0) = \exp(tA)\, x_0 \qquad A \in \mathbb{R}^{n\times n}$
$x_0 \in \mathbb{R}^n$

$\phi_0(x_0) = \exp(0)\, x_0 = I\, x_0 = x_0$ ✓

$\phi_t \circ \phi_s(x_0) = \phi_t(\phi_s(x_0)) = \exp(tA)\, \phi_s(x_0) =$

$= \exp(tA)\exp(sA)\, x_0 = \exp((t+s)A)\, x_0$

$= \phi_{t+s}(x_0)$ ✓

<u>Ex</u>: $x' = 2t$

$x = t^2 + x_0$

$\phi_t(x_0) = t^2 + x_0$

$(\phi_t \circ \phi_s)(x_0) = \phi_t(\phi_s(x_0)) = t^2 + \phi_s(x_0) =$

$= t^2 + s^2 + x_0 \neq (t+s)^2 + x_0 = \phi_{t+s}(x_0)$

it is not a dynamical system

The existence and uniqueness theorem

$X' = F(X) \qquad X(t_0) = X_0$

$X_0 \in \mathbb{R}^n$. Assume $F: \mathbb{R}^n \to \mathbb{R}^n$ is C^1.

(C^1 means the function and all its partial derivatives are continuous).

then, there exists $a > 0$ and a unique solution

$$x : (t_0 - a, t_0 + a) \to \mathbb{R}^n.$$

Ex: $\quad x' = x^2 \qquad x(0) = 1$

$$\frac{dx}{dt} = x^2 \qquad \int \frac{dx}{x^2} = \int dt$$

$-\frac{1}{x} = t + c \qquad t=0 \quad x=1 \Rightarrow c=-1$

$x = \frac{1}{1-t} \qquad F(x) = x^2$ is C^1

Solutions become unbounded if they can not be extended

Th: $F : \mathbb{R}^n \to \mathbb{R}^n$. F is C^1.
x a solution of $x' = F(x)$.
$x : (\alpha, \beta) \to \mathbb{R}^n$. (α, β) is maximal. $x(t)$ does not remain bounded as $t \to \beta$. Same as $t \to \alpha$

If $\alpha, \beta \in \mathbb{R}$.

Continuous dependence on initial conditions

Ex: $x = x_0 e^{at}$ is the sol of

$x' = ax$

$x(0) = x_0$

<u>Th</u>: $x' = F(x)$. $F : \mathbb{R}^n \to \mathbb{R}^n$ C^1

$x : [t_0, t_1] \to \mathbb{R}^n$ is a solution

$x(t_0) = x_0$. Then, there exists $\varepsilon > 0$

and K such that, if $y_0 \in \mathbb{R}^n$

$$y_0 = \begin{bmatrix} y_{01} \\ \vdots \\ y_{0n} \end{bmatrix} \quad x_0 = \begin{bmatrix} x_{01} \\ \vdots \\ x_{0n} \end{bmatrix}$$

if $\sqrt{(y_{01} - x_{01})^2 + \cdots + (y_{0n} - x_{0n})^2} < \varepsilon$,

then $|y(t) - x(t)| \le |y_0 - x_0| e^{K(t - t_0)}$

$y' = F(y)$ $y(0) = y_0$

$|a| = \sqrt{a_1^2 + \cdots + a_n^2}$ if $a \in \mathbb{R}^n$

Ex: 1) $x' = ax$ $\quad x(t) = x_0 e^{at}$ $\quad t_0 = 0$

$y(t) = y_0 e^{at}$

$$|y(t) - x(t)| = |x_0 - y_0| e^{at}$$

$k = a$

2) $x' = Ax$ $\quad$ A has n l.i. eigenvectors A is symmetric

$$k = \max_{\lambda \text{ eigenvalue of } A} \lambda$$

Reminder: If $A \in \mathbb{R}^{n \times n}$. A is symmetric, then:

1) Each eigenvalue of A is real

2) A has n linearly independent

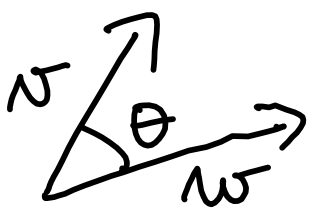
eigenvectors that are orthogonal.

$$A v_i = \lambda_i v_i \qquad 1 \le i \le n$$

$$v_i \neq 0. \qquad v_i \cdot v_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

$$v \cdot w = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

$$v \cdot w = |v||w| \cos\theta$$



If $A \in \mathbb{R}^{n \times n}$ A symmetric. then

$$x' = Ax \quad x = c_1 v_1 e^{\lambda_1 t} + \cdots + c_n v_n e^{\lambda_n t}$$

$$y = d_1 v_1 e^{\lambda_1 t} + \cdots + d_n v_n e^{\lambda_n t}$$

x & y two solutions. $\{v_1, \ldots, v_n\}$ orthonormal.

$$|x-y|^2 =$$

$$\left|(c_1-d_1)v_1 e^{\lambda_1 t} + \cdots\cdots + (c_n-d_n)v_n e^{\lambda_n t}\right|^2$$

$$= (c_1-d_1)^2 e^{2\lambda_1 t} + \cdots + (c_n-d_n)^2 e^{2\lambda_n t} \leq$$

$$\leq \underbrace{\left[(c_1-d_1)^2 + \cdots + (c_n-d_n)^2\right]}_{|x(0)-y(0)|^2} e^{2kt}$$

$t_0 = 0$

$$k = \max_{1 \leq i \leq n} \lambda_i$$

$$|x(t)-y(t)| \leq |x(0)-y(0)|\, e^{k(t-0)}$$

<u>Corollary</u>: $\phi(t,x) = \phi_t(x)$ is the solution of $x' = F(x)$ $x(0) = x$

$$\boxed{\frac{\partial \phi}{\partial t}(t,x) = F(\phi(t,x)) \text{ and } \phi(0,x) = x}$$

ϕ is continuous on both x and t.

<u>Th</u>: $x' = F_a(x)$. Assume $F_a(x)$ is continously differentiable on both x and a

Ex $x' = x^2 + a$

then $\phi(t, x)$ is a continuous function of t, x and a