4541_13

Tuesday, October 2, 2018 9:42 AM

Ex Compute $\exp(A)$

1) $A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$

$$A^2 = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \mu^2 \end{bmatrix}$$

$$A^k = \begin{bmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{bmatrix}$$

$$\exp(A) = \sum_{k=0}^{\infty} \frac{\begin{bmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{bmatrix}}{k!} =$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \end{bmatrix} = \begin{bmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{bmatrix}$$

2) $A = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}$

$$A^0 = I \qquad A = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}$$

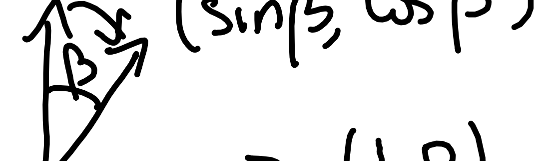
$$A^2 = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} = \begin{bmatrix} -\beta^2 & 0 \\ 0 & -\beta^2 \end{bmatrix}$$

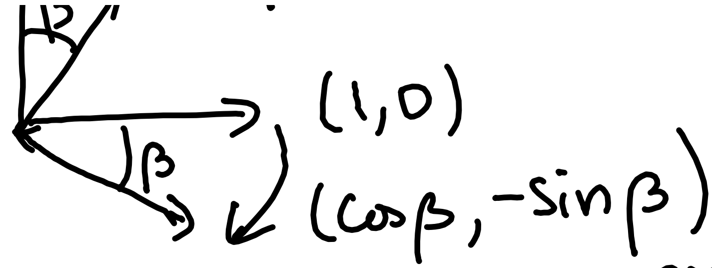
$$A^3 = \begin{bmatrix} 0 & -\beta^3 \\ \beta^3 & 0 \end{bmatrix} \qquad A^4 = \begin{bmatrix} \beta^4 & 0 \\ 0 & \beta^4 \end{bmatrix}$$

$$\begin{bmatrix} 1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \frac{\beta^6}{6!} \cdots & \beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} - \frac{\beta^7}{7!} \cdots \\ -\beta + \frac{\beta^3}{3!} - \frac{\beta^5}{5!} + \frac{\beta^7}{7!} \cdots & 1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \frac{\beta^6}{6!} \cdots \end{bmatrix}$$

$$\begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

rotation clock-wise by β





Prop: $A, B, T \in \mathbb{R}^{n\times n}$ T invertible.

then:

1) $\exp(T^{-1}AT) = T^{-1}(\exp A)T$

why?

$$\sum_{k=0}^{N} \frac{(T^{-1}AT)^k}{k!} = \sum_{k=0}^{N} \frac{T^{-1}A^kT}{k!} = T^{-1}\left(\sum_{k=0}^{N} \frac{A^k}{k!}\right)T$$

$$(T^{-1}AT)^2 = (T^{-1}AT)(T^{-1}AT)$$

take limit $N \to \infty$ to get

$$\exp(T^{-1}AT) = T^{-1}\exp(A)\,T$$

exp [illegible]

2) If $AB = BA$, then $\exp(A+B) =$

$= \exp(A)\exp(B)$

$$(A+B)^k = A^k + \binom{k}{1} A^{k-1}B + \binom{k}{2} A^{k-2}B^2 + \dots$$

$$\dots + \binom{k}{k-1} A B^{k-1} + B^k$$

$$(A+B)(A+B) = A^2 + AB + BA + B^2$$
$$= A^2 + 2AB + B^2$$

if $AB = BA$

using this, it can be proved

$\exp(A+B) = \exp(A)\exp(B)$

3) $\exp(0) = I$

$A^0 = I$ even if $A = [0]$

$e^x = 1 + x + \frac{x^2}{2!} + \dots$ even for $x = 0$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

ii) $I = \exp(-A + A) = \exp(-A)\exp(A)$

$$\exp(-A) = (\exp(A))^{-1}$$

<u>Prop</u>: $Av = \lambda v \quad v \neq 0$. Then

$$(\exp(A))v = e^{\lambda} v$$

<u>proof</u>:

$$\left(\sum_{k=0}^{N} \frac{A^k}{k!}\right) v = \left(\sum_{k=0}^{N} \frac{\lambda^k}{k!}\right) v$$

$$(\exp(A))v = e^{\lambda} v$$

<u>Obs</u>: $A(t) \in \mathbb{R}^{n \times n}$ $A(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{bmatrix}$

$$\frac{dA}{dt} = A' = \begin{bmatrix} \frac{d}{dt}a_{11}(t) & \cdots & \frac{d}{dt}a_{1n}(t) \\ \vdots & & \vdots \end{bmatrix}$$

$$\frac{dA}{dt} = A' = \begin{bmatrix} \vdots & & \vdots \\ \frac{d}{dt}a_{n1}(t) & \dots & \frac{d}{dt}a_{nn}(t) \end{bmatrix}$$

Prop: $\frac{d}{dt}\exp(tA) = A\exp(tA)$

proof:

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty} f_n(t)\right) \overset{?}{=} \sum_{n=0}^{\infty} \frac{d}{dt} f_n(t)$$

not always

It is true at $t \in [a,b]$ if

$$M_{a,b}^{N} = \max_{t\in[a,b]} \left|\sum_{n=N+1}^{\infty} f_n(t)\right|$$

If $M_{a,b}^{N} \xrightarrow[N\to\infty]{} 0$ then

$$\frac{d}{dt}\left(\sum_{n=0}^{\infty} f_n(t)\right) = \sum_{n=0}^{\infty} \frac{d f_n(t)}{dt} \quad \text{for all}$$

$$\frac{d}{dt}\left(\sum_{n=0} f_n(t)\right) = \sum_{n=0} \frac{d f_n}{dt}$$

$t \in [a,b]$.

$$\frac{d}{dt} \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^k =$$

$$= A \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} A^{k-1} = A \exp(tA)$$

<u>Th</u>: $A \in \mathbb{R}^{n \times n}$. The solution of the IVP

$$\begin{aligned} x' &= Ax \\ x(0) &= x_0 \end{aligned} \quad \text{is}$$

$$x(t) = \exp(tA)\, x_0$$

<u>proof</u>: $x(0) = \exp(0)\, x_0 = I x_0 = x_0$

$$x'(t) = \frac{d}{dt}\left[\exp(tA)\, x_0\right] = \left[\frac{d}{dt}\exp(tA)\right] x_0.$$

$$= A \underbrace{\exp(tA)x_0}_{=x} = Ax$$

<u>Ex.</u> Solve $x' = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} x \quad x(0)=x_0$

$$\left(t\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\right)^2 = t^2 \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = t^2 \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$$

$$\left(t\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\right)^3 = t^3 \lambda \begin{bmatrix} \lambda & 2 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = t^3 \lambda^2 \begin{bmatrix} \lambda & 3 \\ 0 & \lambda \end{bmatrix}$$

$k \geq 1$

$$\left(t\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\right)^k = t^k \lambda^{k-1} \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}$$

$$\left(t\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\right)^{k+1} = \left(t\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\right)^k t\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} =$$

$$= t^k \lambda^{k-1} \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix} t \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = t^{k+1}\lambda^k \begin{bmatrix} \lambda & k+1 \\ 0 & \lambda \end{bmatrix} \checkmark$$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k \lambda^k}{k!} & \sum_{k=1}^{\infty} t^k \frac{\lambda^{k-1} k}{k!} \\ 0 & \sum_{k=0} \frac{t^k \lambda^k}{k!} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

Solution

$$x(t) = \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} =$$

$$= \begin{bmatrix} x_{01} e^{\lambda t} + x_{02} t e^{\lambda t} \\ x_{02} e^{\lambda t} \end{bmatrix}$$

Non-autonomous linear systems

$$x' = Ax + G(t)$$

constant coefficient

<u>Ex</u>: $x'' + bx' + kx = f(t)$

$x_1 = x \qquad x_1' = x_2$

$x_2 = x' \qquad x_2' = -kx_1 - bx_2 + f(t)$

$$x' = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} x + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$$