4541_12

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$x' = Ax$

$\lambda_1, \ldots, \lambda_r$ are the eigenvalues of A

$n_1, \ldots, n_r$ are the multiplicities

$v_1^{(i)}, \ldots, v_{n_i}^{(i)}$ are n_i linearly indepen-dent solutions of $(A-\lambda_i I)^{n_i} v = 0$

$$x_\ell^{(i)} = e^{\lambda_i t} \sum_{k=0}^{n_i-1} \frac{t^k}{k!} (A-\lambda_i I)^k v_\ell^{(i)}$$

$x_\ell^{(i)}$ $(1 \le i \le r, \quad 1 \le \ell \le n_i)$ are n solutions of $x' = Ax$.

$x_\ell^{(i)}(0) = v_\ell^{(i)}$

$v_1^{(1)}, \ldots, v_{n_1}^{(1)}, v_1^{(2)}, \ldots, v_{n_2}^{(2)}, \ldots, v_1^{(r)}, \ldots, v_{n_r}^{(r)}$

are linearly independent.

Def: We say that the system $x' = Ax$ is hyperbolic if all the eigenvalues of A have real part different than zero. In this case, we also say that A is hyperbolic.

Def: Assume $x' = Ax$ is hyperbolic. Assume $Re(\lambda_i) < 0$ if $1 \le i \le s$ and $Re(\lambda_i) > 0$ if $s < i \le r$.

$$S = \text{span}\left\{ v_1^{(1)}, \dots, v_{n_1}^{(1)}, \dots\dots, v_1^{(s)}, \dots, v_{n_s}^{(s)} \right\}$$

(If $Im(\lambda_j) \neq 0$, replace $v_i^{(j)}$ by its real and its imaginary part)

$$U = \text{span}\left\{ v_1^{(s+1)}, \dots, v_{n_{s+1}}^{(s+1)}, \dots, v_1^{(r)}, \dots, v_{n_r}^{(r)} \right\}$$

S is called the stable subspace

U is called the unstable subspace

<u>Obs</u>: If x is a solution, and $x(0) \in S \Rightarrow x(t) \to 0$ as $t \to \infty$

<u>Example</u>:

$$x_1' = x_1 + 2x_2 - x_3$$
$$x_2' = 3x_2 - 2x_3$$
$$x_3' = 2x_2 - 2x_3$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{bmatrix}$$

$$P(\lambda) = (1-\lambda)\left((3-\lambda)(-2-\lambda) + 4\right) =$$
$$= (1-\lambda)\left(\lambda^2 - \lambda - 2\right) = (1-\lambda)(\lambda-2)(\lambda+1)$$

$$\lambda_1 = -1 \quad A - (-1)I = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 4 & -2 \\ 0 & 2 & -1 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$\lambda_2 = 1 \quad A - I = \begin{bmatrix} 0 & 2 & -1 \\ 0 & 2 & -2 \\ 0 & 2 & -3 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_3 = 2 \quad A - 2I = \begin{bmatrix} -1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix} \quad v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$$X = c_1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} e^{2t}$$

S = stable subspace = span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

U = unstable subspace = span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$.

0 is a like a saddle. Page 110

<u>Ex</u>: $x_1' = x_2$

$x_2' = -x_1$

$x_3' = -x_3$

$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

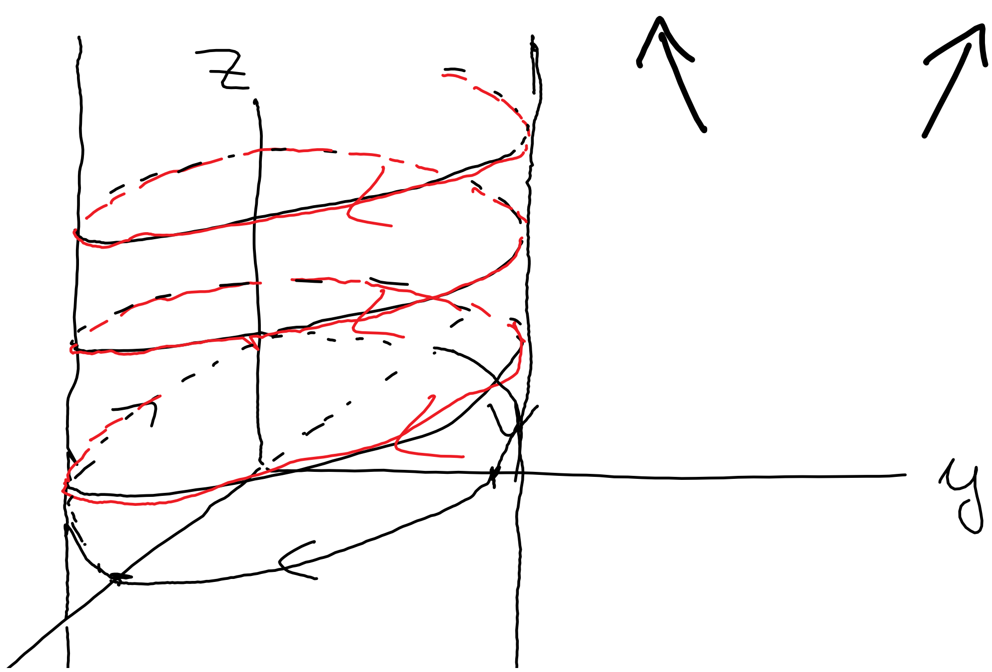
$$P(\lambda) = (\lambda^2+1)(-1-\lambda)$$

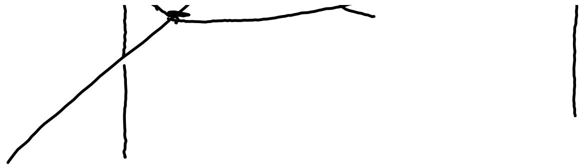
$\lambda = -1$ $\quad A+I = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = i$ $\quad A - iI = \begin{bmatrix} -i & 1 & 0 \\ -1 & -i & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\quad v = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$

$$e^{it} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \\ 0 \end{bmatrix} + i \begin{bmatrix} \sin t \\ \cos t \\ 0 \end{bmatrix}$$

$$X = c_1 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ -\sin t \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \sin t \\ \cos t \\ 0 \end{bmatrix}$$





X

Example $x' = Ax \qquad A = \begin{bmatrix} -0.1 & 0 & 1 \\ -1 & 1 & -1.1 \\ -1 & 0 & -0.1 \end{bmatrix}$

$$P(\lambda) = (-\lambda - 0.1)\left[(1-\lambda)(-0.1-\lambda)\right] +$$

$$\left[1-\lambda\right] = (1-\lambda)\left[(\lambda+0.1)^2 + 1\right]$$

$\lambda = 1 \qquad \lambda = -0.1 \pm i$

$\boxed{\lambda = 1} \quad A - I = \begin{bmatrix} -1.1 & 0 & 1 \\ -1 & 0 & -1.1 \\ 1 & 0 & -1.1 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{\lambda_2 = -0.1 + i} \quad A - (-0.1+i)I =$

$$\begin{bmatrix} -i & 0 & 1 \\ -1 & 1.1-i & -1.1 \\ -1 & 0 & -i \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ i \\ i \end{bmatrix}$$

$\ldots 0.1t \begin{bmatrix} \cos t \end{bmatrix} \quad \begin{bmatrix} \sin t \end{bmatrix}$

$$e^{(-0.1+i)t}\begin{bmatrix}1\\i\\i\end{bmatrix} = e^{-0.1t}\begin{bmatrix}\cos t\\-\sin t\\-\sin t\end{bmatrix} + i\begin{bmatrix}\sin t\\\cos t\\\cos t\end{bmatrix}$$

$$x = c_1\begin{bmatrix}0\\1\\0\end{bmatrix}e^{t} + c_2 e^{-0.1t}\begin{bmatrix}\cos t\\-\sin t\\-\sin t\end{bmatrix} + c_3 e^{-0.1t}\begin{bmatrix}\sin t\\\cos t\\\cos t\end{bmatrix}$$

$$S = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

$$U = \text{span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

<u>Def</u>: Let $A \in \mathbb{R}^{n\times n}$

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

where $A^0 = I$

What does $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ mean?

What does $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ mean?

$$S^{(N)} = \sum_{k=0}^{N} \frac{A^k}{k!} = \begin{bmatrix} S_{11}^{(N)} & \cdots & S_{1n}^{(N)} \\ \vdots & & \vdots \\ S_{n1}^{(N)} & \cdots & S_{nn}^{(N)} \end{bmatrix}$$

$$\sum_{k=0}^{\infty} \frac{A^k}{k!} = \lim_{N\to\infty} S^{(N)} = \begin{bmatrix} \lim_{N\to\infty} S_{11}^{(N)} & \cdots & \lim_{N\to\infty} S_{1n}^{(N)} \\ \vdots & & \vdots \\ \lim_{N\to\infty} S_{n1}^{(N)} & \cdots & \lim_{N\to\infty} S_{nn}^{(N)} \end{bmatrix}$$

when those limit exist.

<u>Prop</u>: $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ converges (same as exists) for all $A \in \mathbb{R}^{n\times n}$

<u>proof</u>: $[A^k]_{ij}$ = ij entry of A^k

$$[A^2]_{ij} = \sum_{l=1}^{n} a_{il} a_{lj}$$

$[A^3]$

$$[A^3]_{ij}$$