Steps to find n solutions of $x' = Ax$ whose initial conditions are linearly independent.

<u>Step 1</u>: Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$ and their algebraic multiplicities, $n_1, n_2, \ldots, n_r$ $\quad (\lambda_i \neq \lambda_j$ if $i \neq j)$.

<u>Step 2</u>: For each eigenvalue λ_i find n_i linearly independent solutions of

$$(A - \lambda_i I)^{n_i} v = 0$$

Call these solutions $v_1^{(i)}, \ldots, v_{n_i}^{(i)}$

<u>Step 3</u>: For each eigenvalue λ_i, set

$$x_\ell^{(i)} = e^{\lambda_i t} \sum_{k=0}^{n_i - 1} \frac{t^k}{k!} (A - \lambda_i I)^k v_\ell^{(i)}$$

$1 \leq \ell \leq n_i \quad , \quad 1 \leq i \leq r$

These $x_e^{(i)}$ are n solutions of $x' = Ax$ whose initial conditions are linearly independent

Step 4 If some eigenvalues are complex, keep only one per pair of complex conjugates. Then take real and imaginary part of any solution left.

Step 5: The general solution is a linear combination of the n solutions you have after step 4

Step 6: Impose the initial conditions to get the constants from Step 5

Fact: Let λ be an eigenvalue of

A with algebraic multiplicity m.

Let v be a non-zero solution of

$(A-\lambda I)^m v = 0$. Let

$$x = e^{\lambda t}\left(\sum_{k=0}^{m-1} \frac{t^k}{k!}(A-\lambda I)^k v\right)$$

then $x' = Ax$

<u>proof</u>:

$$x' = \lambda e^{\lambda t}\left(\sum_{k=0}^{m-1} \frac{t^k}{k!}(A-\lambda I)^k v\right) +$$

$$e^{\lambda t} \sum_{k=1}^{m-1} \frac{t^{k-1}}{(k-1)!} (A-\lambda I)^k v$$

$$Ax = e^{\lambda t} \sum_{k=0}^{m-1} \frac{t^k}{k!} A(A-\lambda I)^k v =$$

$$Ax = e^{\lambda t} \sum_{k=0}^{m-1} \frac{t^k}{k!} (A-\lambda I)^{k+1} v + \lambda e^{\lambda t} \sum_{k=0}^{m-1} \frac{t^k}{k!} (A-\lambda I)^{k} v$$

Examples 1) Solve

$$x_1' = -x_1 + x_2 \qquad x_1(0) = 1$$

$$x_2' = 3x_2 \qquad x_2(0) = 2$$

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} \qquad \lambda_1 = -1 \quad n_1 = 1 \qquad \lambda_2 = 3 \quad n_2 = 1$$

$$\boxed{\lambda_1 = -1} \qquad (A - \lambda_1 I)^{n_1} = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}$$

$$(A - \lambda I)^{n_1} v = 0 \qquad \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{\lambda_2 = 3} \qquad (A - \lambda_2 I)^{n_2} = \begin{bmatrix} -4 & 1 \\ \end{bmatrix}$$

$\boxed{\lambda_2 = 3}$ $(A-\lambda_2 I)^{n_2} = \begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix}$

$v = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

λ	mult	vectors
-1	1	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
3	1	$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$x = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Set $t=0$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$c_2 = \frac{1}{2}$

$c_1 = \frac{1}{2}$

$$x = \frac{1}{2} e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{2} e^{3t} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$x_1 = \frac{1}{2} e^{-t} + \frac{1}{2} e^{3t}$$

$x_2 = 2e^{3t}$

Example: $x_1' = x_1 - 4x_2$

$x_2' = x_1 + x_2$

$x_3' = -2x_3$

$$A = \begin{bmatrix} 1 & -4 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$P(\lambda) = \det \begin{bmatrix} 1-\lambda & -4 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda)(-2-\lambda) -$$

$$-(-4)\,1(-2-\lambda) = (-2-\lambda)\left[(1-\lambda)^2 + 4\right] = 0$$

$\lambda_1 = -2 \qquad \lambda_2 = 1+2i \qquad \lambda_3 = 1-2i$

$\boxed{\lambda_1 = -2}$ $\quad A-(-2)I = \begin{bmatrix} 3 & -4 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\boxed{\lambda_2 = 1+2i}$ $\quad A-(1+2i)I = \begin{bmatrix} -2i & -4 & 0 \\ 1 & -2i & 0 \\ 0 & 0 & -3-2i \end{bmatrix}$

$\begin{bmatrix} 2i \end{bmatrix}$

$$v = \begin{bmatrix} 2i \\ 1 \\ 0 \end{bmatrix}$$

$$x_1 = e^{-2t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad x_2 = e^{(1+2i)t}\begin{bmatrix} 2i \\ 1 \\ 0 \end{bmatrix} =$$

$$= e^{t} e^{2it}\begin{bmatrix} 2i \\ 1 \\ 0 \end{bmatrix} = e^{t}\begin{bmatrix} 2i\,e^{2it} \\ e^{2it} \\ 0 \end{bmatrix} =$$

$$= e^{t}\begin{bmatrix} 2i(\cos 2t + i \sin 2t) \\ \cos 2t + i \sin 2t \\ 0 \end{bmatrix} = e^{t}\begin{bmatrix} 2i\cos(2t) - 2\sin(2t) \\ \cos(2t) + i\sin(2t) \\ 0 \end{bmatrix}$$

$$= e^{t}\begin{bmatrix} -2\sin(2t) \\ \cos(2t) \\ 0 \end{bmatrix} + i\, e^{t}\begin{bmatrix} 2\cos(2t) \\ \sin(2t) \\ 0 \end{bmatrix}$$

General solution

$$x = c_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_2\begin{bmatrix} -2\sin(2t) \\ \cos(2t) \\ 0 \end{bmatrix} e^{t} + c_3\begin{bmatrix} 2\cos(2t) \\ \sin(2t) \\ 0 \end{bmatrix} e^{t}$$

<u>Example</u>: $x_1' = 2x_1 + x_2$

$x_2' = 2x_2 + x_3$

Example: $x_1' = 2x_1 + x_2$

$x_2' = 2x_2 + x_3$

$x_3' = x_3$

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Eigenvalue	muliplicity
2	2
1	1

$\boxed{\lambda_1 = 1}$ $A - I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\boxed{\lambda_2 = 2}$ $\boxed{n_2 = 2}$ $(A - 2I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}^2 =$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are two linearly independent solutions

of $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} v = 0.$

$v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$ $\boxed{\lambda = 2}$ $\boxed{m = 2}$ then $\begin{bmatrix} 0 \end{bmatrix}$

$$x_1 = \begin{bmatrix}1\\-1\\1\end{bmatrix} e^t \qquad \boxed{\lambda = 2} \qquad \boxed{m = 2} \qquad \text{then } v = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$x = e^{\lambda t} \sum_{k=0}^{m-1} \frac{t^k}{k!} (A - \lambda I)^k v$$

$$x_2 = e^{2t} \left\{ \begin{bmatrix}1\\0\\0\end{bmatrix} + t \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & -1\end{bmatrix} \begin{bmatrix}1\\0\\0\end{bmatrix} \right\} = \begin{bmatrix}1\\0\\0\end{bmatrix} e^{2t}$$

$$x_3 = e^{2t} \left\{ \begin{bmatrix}0\\1\\0\end{bmatrix} + t \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & -1\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \right\} =$$

$$x_3 = e^{2t} \left(\begin{bmatrix}0\\1\\0\end{bmatrix} + t \begin{bmatrix}1\\0\\0\end{bmatrix} \right)$$

the general solution is

$$x = c_1 e^t \begin{bmatrix}1\\-1\\1\end{bmatrix} + c_2 \begin{bmatrix}1\\0\\0\end{bmatrix} e^{2t} + c_3 e^{2t} \begin{bmatrix}t\\1\\0\end{bmatrix}$$