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$\mathbb{R}^{n\times n}$ = set of n by n matrices

I = identity matrix = $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$

$A \in \mathbb{R}^{n\times n}$ has an inverse or is invertible if there exists $A^{-1} \in \mathbb{R}^{n\times n}$ such that

$$A A^{-1} = A^{-1} A = I$$

<u>Obs</u>: A has an inverse if and only if the columns of A are linearly independent if and only if $\det(A) \neq 0$

<u>Homogeneous</u> <u>systems</u> <u>of</u> <u>1st</u> <u>order</u> <u>linear</u> <u>constant</u> <u>coefficients</u> <u>differential</u> <u>equations</u>

equations

$$x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

$$x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$

$$\vdots \qquad \vdots \qquad \qquad \vdots$$

$$x_n' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

$a_{ij} \in \mathbb{R}$, known

Goal: find $x_i(t)$ $\quad 1 \le i \le n$

Matrix notation

$$x = x(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$x' = Ax$$

$\underline{\text{Obs}}$: Let $x = e^{\lambda t} v$ $\quad v \in \mathbb{R}^n$.

$$\left.\begin{array}{l} x' = \lambda e^{\lambda t} v \\ Ax = e^{\lambda t} A v \end{array}\right\} \Rightarrow x \text{ is a solution of } x' = Ax \text{ if and only if}$$

$$\boxed{\lambda v = A v}$$

$\underline{\text{Fact}}$: 1) Let $u \in \mathbb{R}^n$. There exists a unique solution of

$$(\text{IVP}) \begin{cases} x' = Ax \\ x(0) = u \end{cases}$$

2) If $x_1, \ldots, x_r$ are solutions of $x' = Ax$, and $c_1, \ldots, c_r \in \mathbb{R}$, then $c_1 x_1 + \cdots + c_r x_r$ is also a solution of $x' = Ax$

3) If $v_1, \ldots, v_n \in \mathbb{R}^n$ are linearly

independent, then $v_1, \ldots, v_n$ are a basis of $\mathbb{R}^n$. Thus, any $u \in \mathbb{R}^n$ can be written in a unique way as a linear combination of $v_1, \ldots, v_n$.

4) If $x_1, \ldots, x_n$ are n solutions of $x' = Ax$ and $x_1(0), \ldots, x_n(0)$ are linearly independent, then x is a solution of $x' = Ax$ if and only if x is a linear combination of $x_1, x_2, \ldots, x_n$

__Goal__: Find n solutions of $x' = Ax$ whose initial values are linearly independent.

__Obs__: Eigenvectors with different eigenvalues are linearly independent.

Obs: a & λ numbers $a, \lambda \in \mathbb{C}$

$x' = ax$ $\quad x = x(t)$ $\quad v \in \mathbb{C}$

$$x = e^{at} v = e^{(a-\lambda)t} e^{\lambda t} v =$$

$$= e^{\lambda t} \left(\sum_{k=0}^{\infty} \frac{(a-\lambda)^k}{k!} t^k \right) v =$$

$$= e^{\lambda t} \left[\sum_{k=0}^{\infty} \frac{t^k}{k!} (a-\lambda)^k v \right]$$

Replace a by A

$$x = e^{\lambda t} \left[\sum_{k=0}^{\infty} \frac{t^k}{k!} (A-\lambda I)^k v \right]$$

Obs: If $(A-\lambda I)v = 0$, then

$$(A-\lambda I)^k v = (A-\lambda I)^{k-1} \underbrace{(A-\lambda I) v}_{=0} = 0$$

as long as $k \geq 1$

$= 0$

as long as $k \geq 1$

<u>th</u>: Let $A \in \mathbb{R}^{n\times n}$. Let $P(\lambda) = \det(A - \lambda I) =$

$= (-1)^n (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_r)^{n_r}$

where $\lambda_i \neq \lambda_j$ if $i \neq j$, $\lambda_i \in \mathbb{C}$, $n_i \geq 1$

$n_1 + n_2 + \cdots + n_r = n$.

n_i is called the algebraic multiplicity of λ_i

$\dim\left(\{v \in \mathbb{R}^n : (A - \lambda_i I)v = 0\}\right)$ is called

the geometric multiplicity of λ_i.

For each i,

$1 \leq$ geometric multiplicity of $\lambda_i \leq$ algebraic multiplicity of λ_i.

For each i, there are n_i linearly independent solutions of

$$(A - \lambda_i I)^{n_i} v = 0$$

$$(A-\lambda_i I) \, v = 0$$

<u>Example:</u> $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$P(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} = -(\lambda-1)^2(\lambda-2)$$

$\lambda_1 = 1 \qquad \lambda_2 = 2$

Eigenvalue	alg mult	geo mult
1	2	1
2	1	1

$\lambda_1 = 1 \quad A - 1I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 2 \quad A - 2I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 1 \qquad (\quad)^2$

$\lambda_1 = 1$

alg mult $= 2$

$$(A - 1I)^2 v = 0$$

$$(A - I)^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(A - I)^2 v = 0$ $\qquad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are two linearly independent solutions.