Differential equations

These are equations where the unknown is a function and the derivatives of the function appear in the equation.

Notatation: $x = x(t)$ $x' = \frac{dx}{dt}$

$\boxed{1.1}$ $x' = ax$

all the solutions of $x' = ax$ are of the form $x = ke^{at}$

For each k, we have a solution. We call this collection the general solution

Initial condition: Give $x(t_0)$ for some t_0. $x(t_0) = u$ u given

Initial value problem

Initial value problem

$$x' = a x$$
$$x(t_0) = u$$

$x = k e^{at}$ impose the initial conditions to get $x = u\, e^{a(t-t_0)}$

Note: there is a unique solution of the initial value problem

Def: An equilibrium point or equilibrium solution for a differential equation is a solution that is a constant.

Note: the only equilibrium point for $x' = ax$ is $x = 0$ (corresponds to $k=0$).

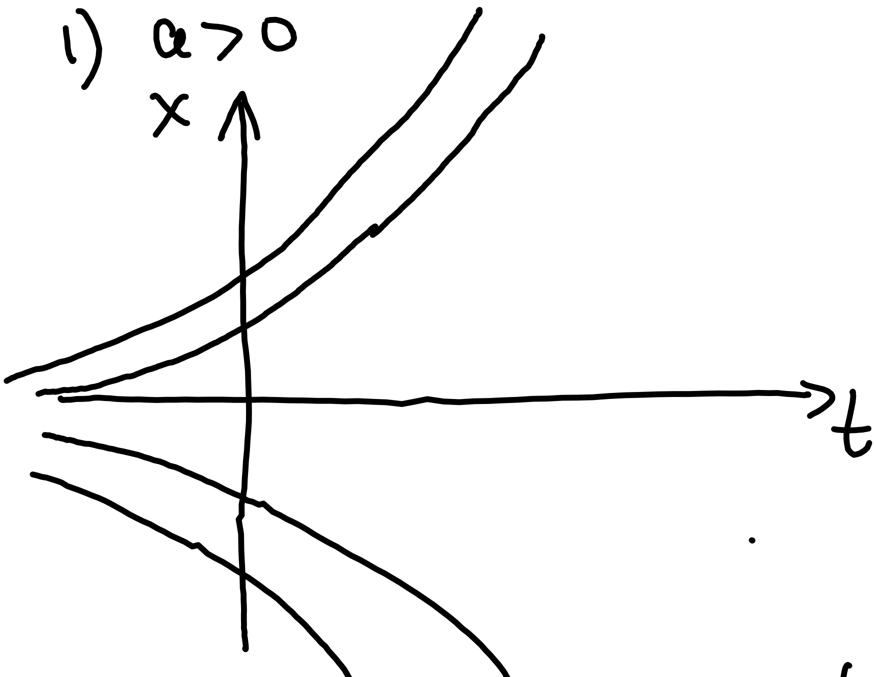
Parameters in equations: a in $x' = ax$

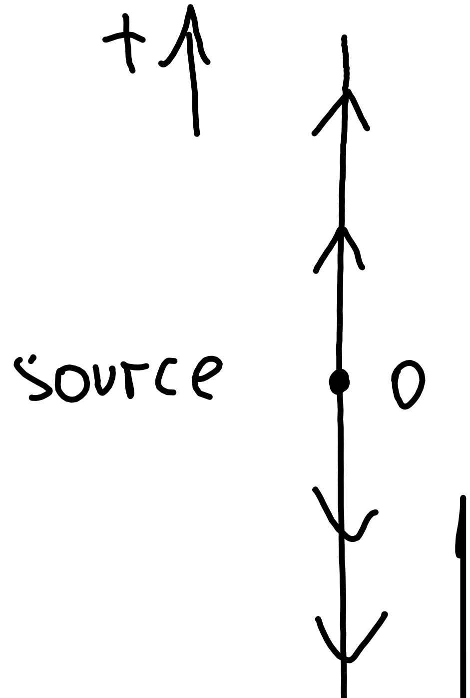
Parameters in equations: a in $x' = ax$ is a parameter.

Goal: Understand how the solutions of $x' = ax$ change as a changes.

Obs: $x = ke^{at}$. then

1) $a > 0$

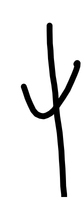


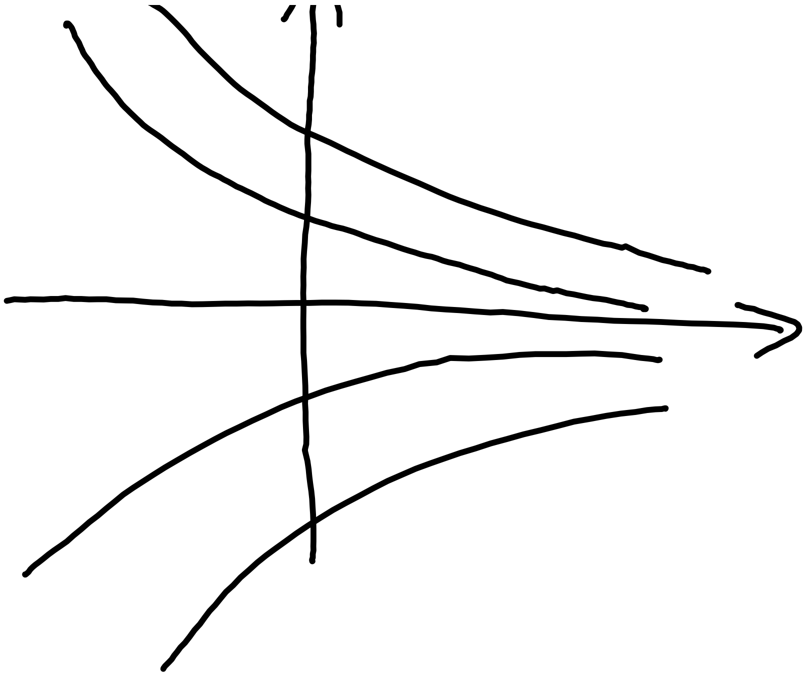


$$\lim_{t \to \infty} x(t) = \begin{cases} +\infty & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ -\infty & \text{if } k < 0 \end{cases}$$

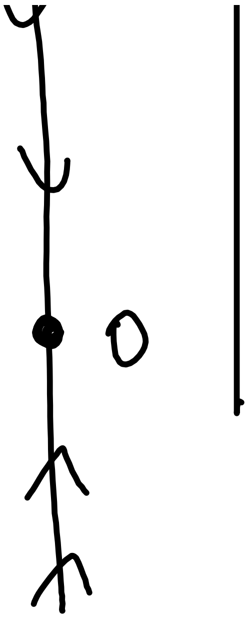
phase line

2) $a < 0$



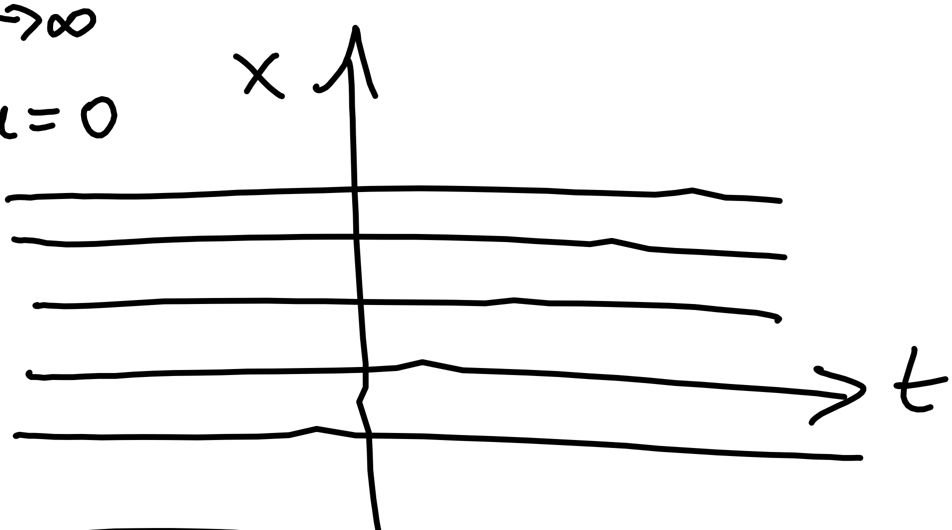


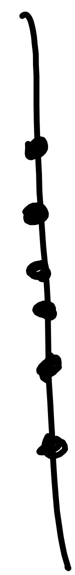
Sink



$\lim_{t\to\infty} x(t) = 0$

3) $a = 0$





$\boxed{\text{Phase line}}$ 1) think of $x(t)$ as the position of a particle at time t.

2) the particle moves on a straight line

3) the equilibrium solution are plotted as a solid dot

4) the arrows indicate the direction in which the particle moves.

Obs: the equation $x' = ax$ is stable for $a \neq 0$. This means that if a changes a little, the qualitative behavior of the solutions do not change. (we just have to make sure that the sign of a does not change).

If $a=0$, any change of a (no matter how small the change), changes the qualitative behavior of the solutions.

We say that we have a bifurcation at $a=0$.

$\boxed{1.2}$ $x' = a x (1-x)$ logistic growth model

$a > 0$

x = size of a population

If $0 < x < 1$, then $x' > 0$, then x incre-
ases

If $x > 1$, then $x' < 0$, then x decreases

<u>First order - non-linear - autonomous differential equations</u>

these are equations of the form

$$x' = f(x)$$

for some known function f.

<u>Example</u> $x' = \underbrace{a x (1-x)}$

$$\underbrace{\qquad\qquad}_{f(x)}$$

<u>Obs</u>: the unique solution of

$$x' = a\,x(1-x)$$

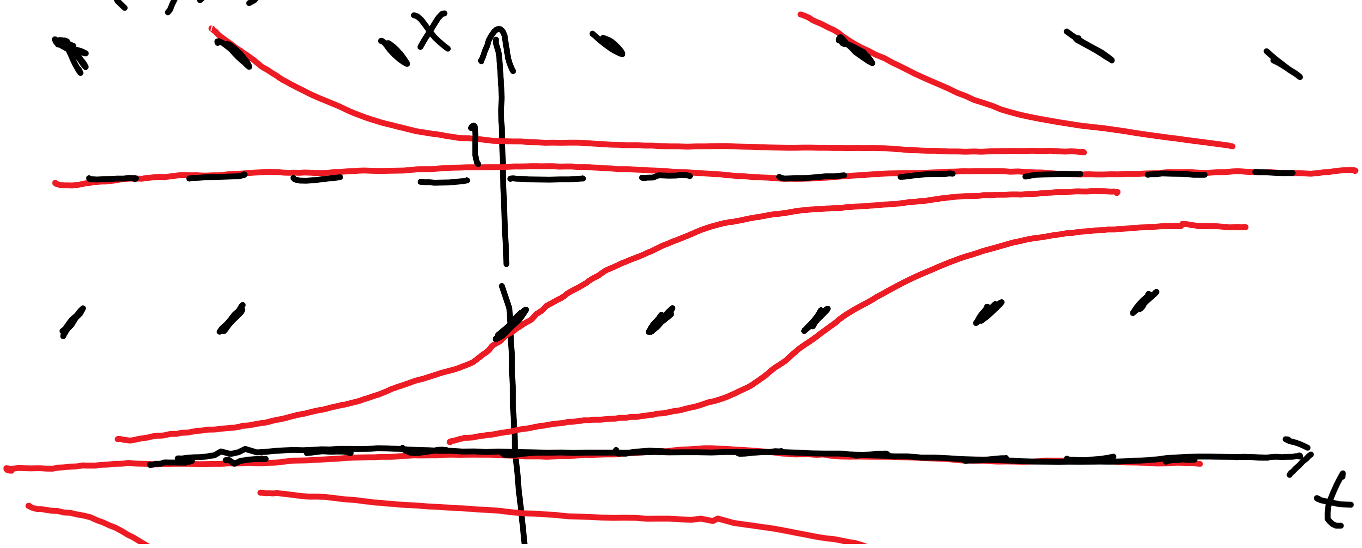
$$x(0) = u$$

is

$$x = \frac{u}{u + (1-u)\,e^{-at}}$$

Slope field

Plot at (t,x) a little segment of slope $a\,x(1-x)$. Do this for many (t,x). Set $a=1$



~~an~~ solution of $x' = a\,x(1-x)$ is going to be tangent to these little segments