Def: $A \amalg B \Leftrightarrow A \cup B$ when $A \cap B = \emptyset$.

<u>Theorem</u> $S \subset E$. S is not connected

$\Leftrightarrow S \subset A \amalg B$ with A, B open sets in E.

and $A \neq \emptyset$ & $B \neq \emptyset$.

$(A \amalg B = A \cup B$ when $A \cap B = \emptyset)$

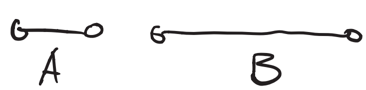
<u>proof</u> $\Leftarrow)$ A, B open $\Rightarrow$

$A \cap S$ and $B \cap S$ open in S

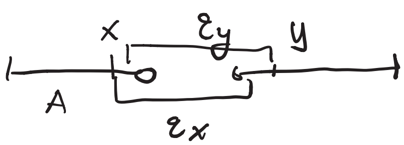
$S = (A \cap S) \amalg (B \cap S)$

and $A \cap S \neq \emptyset$ & $B \cap S \neq \emptyset$

$\Rightarrow S$ is not connected.



$\Rightarrow)$ $\exists\ A'$ & B' open in S such that $A' \amalg B' = S$.



$\forall x \in A'$, $\exists\ \varepsilon_x$ s.t. $B^S_{\varepsilon_x}(x) \subset A'$

Let $A = \bigcup_{x \in A'} B^E_{\frac{\varepsilon_x}{2}}(x)$.

$\forall y \in B'$, $\exists\, \varepsilon_y$ s.t. $B^S_{\varepsilon_y}(y) \subset B'$

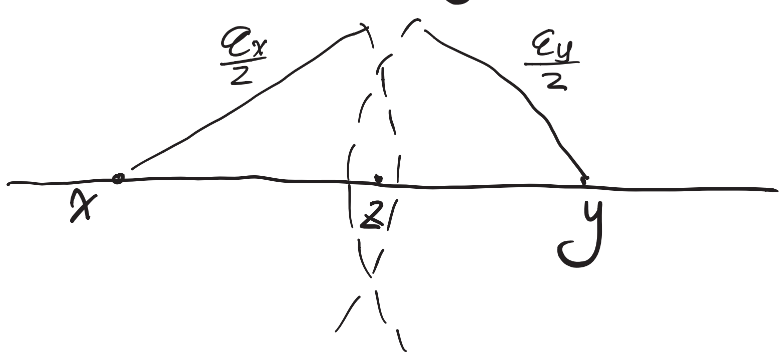
Let $B = \bigcup_{y \in B'} B^E_{\frac{\varepsilon_y}{2}}(x)$.

Claims: 1) $A \cap S = A'$ $\quad B \cap S = B'$

2) $A \cap B = \emptyset$

If $z \in A \cap B \Rightarrow \exists\, x \in A'$ & $y \in B'$, s.t.

$z \in B^E_{\frac{\varepsilon_x}{2}}(x) \cap B^E_{\frac{\varepsilon_y}{2}}(y)$



so, $d(x,y) \le d(x,z) + d(z,y)$

$< \frac{\varepsilon_x}{2} + \frac{\varepsilon_y}{2}$

$\le \max(\varepsilon_x, \varepsilon_y)$

$= \varepsilon_y \quad$ (assume $\varepsilon_x < \varepsilon_y$)

then $x \in B_{\varepsilon_y}(y) \cap S \subset B'$ but $x \in A'$ and $A' \cap B' = \emptyset$. Contradiction.

<u>Theorem</u> (E, d), $S_i \subset E$, $i \in I$, Each S_i connected. Assume $\exists\, i_0 \in I$, s.t. $S_i \cap S_{i_0} \neq \emptyset$ $\forall\, i \in I$. Then, $S = \bigcup_{i \in I} S_i$ is connected

<u>proof</u>: Let A, B be two open sets in E. such that $\bigcup_{i \in I} S_i \subset A \amalg B$.

Since S_{i_0} is connected and $S_{i_0} \subset A \amalg B$ then $S_{i_0} \subset A$ or $S_{i_0} \subset B$.

Assume $S_{i_0} \subset A$, Let $i \in I$ then $S_i \subset A$ or $S_i \subset B$.

Let $x \in S_{i_0} \cap S \neq \emptyset$, then, if $S_i \subset B$ we would have $x \in S_{i_0} \cap S_i \subset A \cap B = \emptyset$ it is impossible $\Rightarrow S_i \subset A$ $\forall\, i \in I$.

<u>Theorem</u> Let $I \subset \mathbb{R}$. I is an interval. I is connected.

~~proof~~ Let A, B be open in $\mathbb{R}$ such that $I \subset A \amalg B$. If $a \in I \cap A$ & $b \in I \cap B$.

Let $S = \{ c \geq a : [a, c) \subset A \}$. $c \leq b$. Indeed, if $b < c$, $b \in [a, c)$ but $b \notin A$ because $b \in B$ & $A \cap B = \phi$. Then S is bounded by b.

Since A is open and $a \in A \Rightarrow \exists\, \varepsilon > 0$ s.t. $(a-\varepsilon, a+\varepsilon) \subset A \Rightarrow$ $c = a+\varepsilon \in S$ then $S \neq \phi$

Let $\bar{c} = l.u.b.(S)$, since $a, b \in I$ and $a \leq \bar{c} \leq b \Rightarrow$ $c \in I$.

If $\bar{c} \in A \subset I$, since A is open, $\exists\, \varepsilon > 0$ s.t. $(\bar{c}-\varepsilon, \bar{c}+\varepsilon) \subset A$.

$\Rightarrow \; [a, \bar{c}+\varepsilon) = [a, \bar{c}) \cup [\bar{c}, \bar{c}+\varepsilon) \subset A$

If $x \in [a, \bar{c}) \Rightarrow$ since $\bar{c} = l.u.b.(S)$

$\exists \; x < y < \bar{c}$ s.t. $y \in S$.

$\Rightarrow [a, y) \subset A \Rightarrow x \in [a, y) \subset A$.

Since $\bar{c} \in A$, $[a, \bar{c}+\varepsilon) \subset A$ impossible because that would mean $\bar{c}+\varepsilon \in S$ but $\bar{c} = l.u.b(S)$.

If $\bar{c} \in B$, B is open $\Rightarrow \exists \, \varepsilon > 0$, s.t.

$(\bar{c}-\varepsilon, \bar{c}+\varepsilon) \subset B \Rightarrow (\bar{c}-\varepsilon, \bar{c}] \cap S \neq \phi$.

Then $\bar{c}-\varepsilon$ would be an upper bound of S.

That impossible since $\bar{c} > \bar{c}-\varepsilon$ and $\bar{c}$ is l.u.b of S.