4317_1

Tuesday, August 21, 2018 12:08 PM

Set theory

$x \in S$ x belongs to S

$x \notin S$ x does not belong to S.

$\{1, 2, 5\}$

$\{x : \text{statement}\}$

$\{x : x \text{ is an even number}\} = \{\dots, -4, -2, 0, 2, 4, \dots\}$

$X \subset Y$. X is included in Y. Every element of X belongs to Y

$\Rightarrow$ means implies

Ex: $\{\text{positive multiples of } 4\} \subset \{\text{even numbers}\}$

Def: $X \not\subset Y$. X is not included in Y.

$\exists\, x \in X$ such that $x \notin Y$

$\uparrow$

there exists

$X \subset Y$. We say that X is a subset of Y.

X is a proper subset of Y if $X \subset Y$ but $X \neq Y$

Def: $\emptyset$ empty set

1) $X \cap Y = \{x : x \in X \ \& \ x \in Y\}$ intersection

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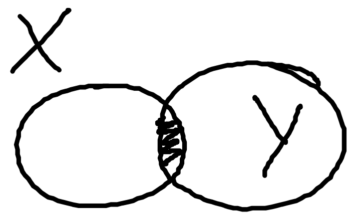
2) $X \cup Y = \{x : x \in X \text{ or } x \in Y\}$ union

3) $X^c = \{x : x \notin X\}$ complement



Example: $A = \{\text{even numbers}\}$

$A^c = ?$

Obs: $(X \cap Y)^c = X^c \cup Y^c$

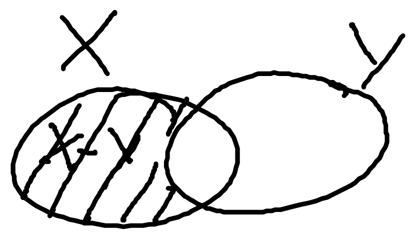
proof: $\boxed{x \in (X \cap Y)^c} \Longleftrightarrow x \notin X \cap Y \Longleftrightarrow$

$x \notin X$ or $x \notin Y \Longleftrightarrow x \in X^c$ or $x \in Y^c$

$\Longleftrightarrow \boxed{x \in X^c \cup Y^c}$

$x \in (X \cap Y)^c \Longleftrightarrow x \in X^c \cup Y^c$

Thus $(X \cap Y)^c = X^c \cup Y^c$



Def: $X - Y = \{x \in X : x \notin Y\}$

Family of sets $\{X_i : i \in I\}$ $\{X_i\}_{i \in I}$

I set of indeces.

Example: $X_\lambda = (\lambda, \infty) = \{x \in \mathbb{R} : x > \lambda\}_{\lambda \in \mathbb{R}}$

$X_i = \{\text{multiples of } i\}$ i positive integer.

$\cap X_i = \{x : x \in X_i$ for all $i \in I\}$

$$\bigcap_{i \in I} X_i = \{x : x \in X_i \text{ for all } i \in I\}$$

$$\bigcup_{i \in I} X_i = \{x : x \in X_i \text{ for an } i \in I\}$$

<u>Example</u>: $\left(\bigcap_{i \in I} X_i\right)^c = \bigcup_{i \in I} X_i^c$

<u>Cartesian product</u>

$$X \times Y = \{(x,y) : x \in X \text{ and } y \in Y\}$$

<u>Example</u>: $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x,y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$

$\mathbb{R}$ = set of real numbers

<u>Functions</u> $\quad f : \underset{\text{domain}}{X} \longrightarrow \underset{\text{range}}{Y}$

$x \in X, \ f(x) \in Y$

<u>Example</u>: 1) $X = \{1, 2, 3, 4, 5\}$

x	$f(x)$
1	2
2	1
3	3
4	1
5	2

$f : X \to X$

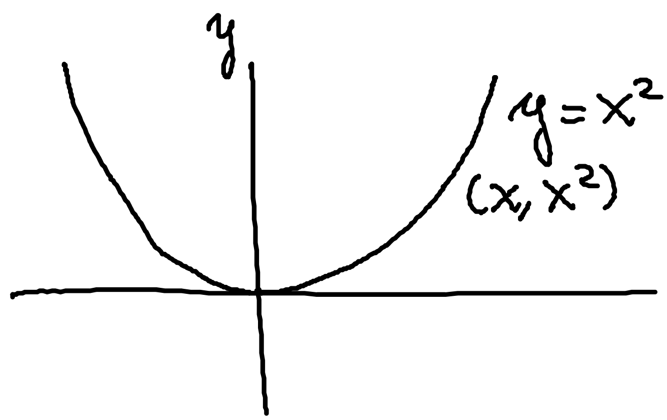
$Y = X$

2) $f : \mathbb{R} \to \mathbb{R} \qquad f(x) = x^2 + 2x$

$\{(x, f(x)) : x \in X\} \subset X \times Y$

2) $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x$ ...

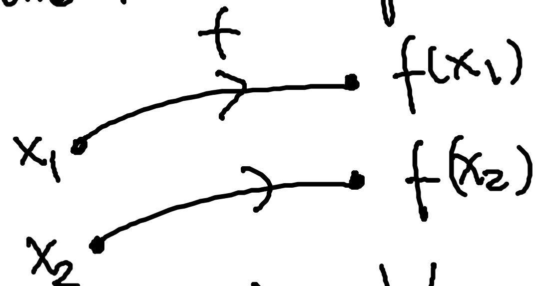
Graph of $f = \{ (x, f(x)) : x \in X \} \subset X \times Y$



Ex: $f(x) = x^2$
$f : \mathbb{R} \to \mathbb{R}$

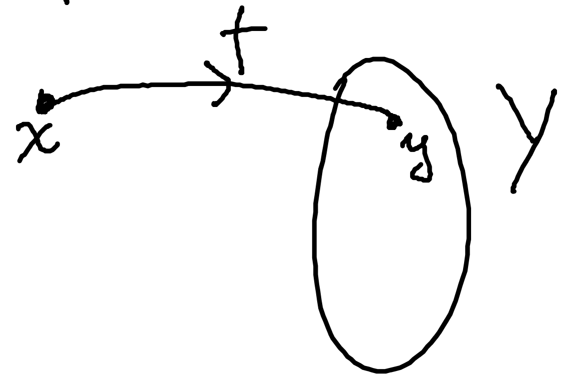
$f : X \to Y$

1) f is one to one if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$



2) f is onto if $\forall y \in Y \; \exists x \in X$ such that $y = f(x)$

for all



<u>Example</u>: 1) $f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$ is not onto

2) $f : \mathbb{R} \to \mathbb{R}_{\geq 0} = \{ y \in \mathbb{R} : y \geq 0 \}$ $\quad f(x) = x^2$ is onto.

<u>Restriction of a function</u>: $f : X \to Y$

Restriction of a function: $f: X \to Y$

$X' \subset X$. Let $g: X' \to Y$ be defined as $g(x') = f(x')$. We call g the restriction of f on X'. We usually denote g by f or $f|_{X'}$

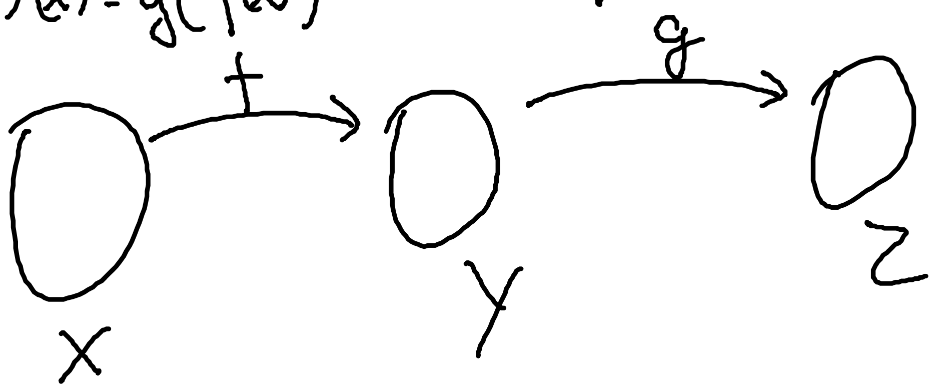
Ex: $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$ is not one to one

$f|_{\mathbb{R}_{\geq 0}}$ is one to one

Composition of functions

$f: X \to Y \qquad g: Y \to Z$

$(g \circ f)(x) = g(f(x))$ composition



one-one = one to one

one-one and onto = one-one onto = one-one correspondance

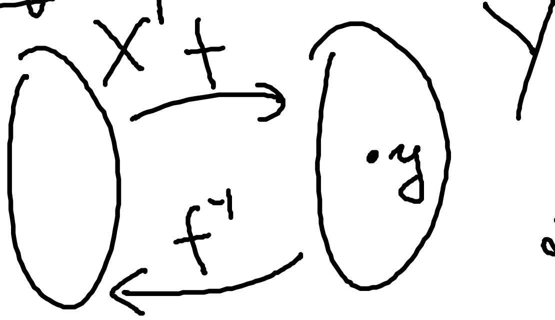
$i_X: X \to X \qquad i_X(x) = x$ is called the identity

Inverses of function

... onto and one to one.

Inverses of function

Def: $f: X \to Y$. f onto and one to one. For each $y \in Y$ $\exists!$ (there exists a unique) $x \in X$ such that $f(x) = y$. We define



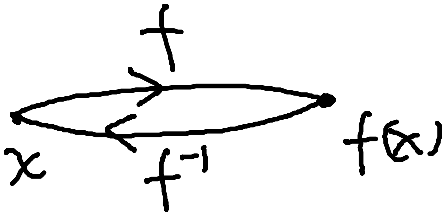
$f^{-1}: Y \to X$ $\quad f^{-1}(y) = x$ if $f(x) = y$

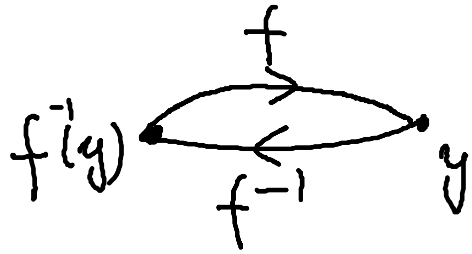
f^{-1} is well define

f^{-1} is called the inverse of f.

Obs: Let $x \in X$. f one to one onto $f: X \to Y$

$f^{-1}(f(x)) = x$. Let $y \in Y$ $\quad f(f^{-1}(y)) = y$





$f^{-1} \circ f = i_X$ $\qquad f \circ f^{-1} = i_Y$

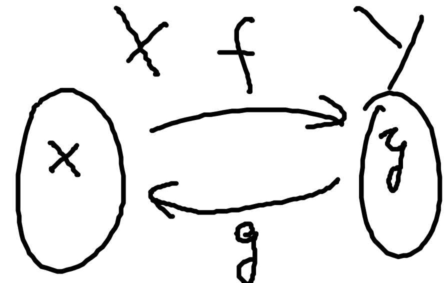
Obs: $(f^{-1})^{-1} = f$

Obs: Let $f: X \to Y$. Then:

1) If $\exists$ $g: Y \to X$ such that $g \circ f = i_X$ then

f is one to one

2) If $\exists\ g: Y \to X$ such that $f \circ g = i_Y$ then f is onto



3) If $\exists\ g_1: Y \to X$ such that $g_1 \circ f = i_X$ and $g_2 : Y \longrightarrow X$ such that $f \circ g_2 = i_Y$ then f is one to one and onto. And $g_1 = g_2 = f^{-1}$

<u>prove</u>: 1) Let $x_1 \in X$, $x_2 \in X$ such that $x_1 \neq x_2$. then $g(f(x_1)) = x_1$ and $g(f(x_2)) = x_2$

$g(f(x_1)) \neq g(f(x_2))$ then $f(x_1) \neq f(x_2)$

The ~~f~~ f is one to one

2) $f(g(y)) = y \quad \forall y$.

Give me y, let $x = g(y)$, then $f(x) = y$.

then f is onto.

3) $g_1 \circ f = i_X \qquad f \circ g_2 = i_Y$

$g_1 = g_1 \circ i_Y = g_1 \circ (f \circ g_2) = (g_1 \circ f) \circ g_2 =$

$i_X \circ g_2 = g_2$

Def: $f: X \to Y$. 1) $X' \subset X$. Then

$f(X') = \{f(x) : x \in X'\}$ image of X'

2) $Y' \subset Y$. Then $f^{-1}(Y') = \{x \in X : f(x) \in Y'\}$

3) $f(X)$ = range or image $\subset Y$ = codomain

Def: 1) positive integers = natural numbers

$= \{1, 2, 3, \dots\} = \mathbb{N}$

2) X is finite if $\exists\, n \in \mathbb{N}$ and

$f : \{1, 2, \dots, n\} \to X$ such that f is

one to one & onto

Example $X = \{1, 2, 7\}$ is finite

$f : \{1, 2, 3\} \to X$ $\quad f(1) = 1$ $\quad f(3) = 7$

$f(2) = 2$

Def: If X is finite, $\#(X)$ = the number of elements of $X = n$ such that $\exists$

$f : \{1, 2, \dots, n\} \to X$ one to one onto.

Can I have $g : \{1, 2, 3, 4\} \to X$

$\quad : \{1, 2, 3\} \to X$

$f: \{1,2,3\} \longrightarrow$

both f & g one to one and onto? No

Let's prove it.

Lemma: If $\exists\ f: \{1,2,..,k\} \to \{1,2,...,n\}$ one to one and onto, then $k=n$.

proof: By induction on n.

$n=1$ $\quad f: \{1,2,...,k\} \to \{1\}$

$f(1)=1 \quad f(2)=1 \cdots\cdots f(k)=1$. But f is one to one. thus $k=1$ because if $k \geq 2$ then $f(1)=f(2)$ contradiction.

$n>1$. $\quad f: \{1,2,..,k\} \longrightarrow \{1,2,..,n\}$

- f one to one and onto.

$n>1$ and since f is onto, $\exists\ x_1, x_2$ such that $f(x_1)=1$ and $f(x_2)=2$ $\quad x_1, x_2 \in \{1,...,k\}$

then $k \geq 2$. then $k>1$.

Let $l \in \{1,..,k\}$ such that $f(l)=n$ (I can find l because f is onto). l is unique because f is one to one

$f' : \{1,\dots,l-1\} \cup \{l+1,\dots,k\} \to \{1,2,\dots,n-1\}$

f' is one-to-one and onto

$$g(i) = \begin{cases} i & \text{if } 1 \le i \le l-1 \\ i+1 & \text{if } l \le i \le k-1 \end{cases}$$

$1 \xrightarrow{g} 1$

$2 \to 2 \xrightarrow{f'} \{1,\dots,n-1\}$

$l-1 \to l-1$

$l \to l+1$

$\vdots$

$k-1 \to k$

g is one-to-one and onto $\Downarrow$ $\Downarrow$

$f' \circ g : \{1,\dots,k-1\} \to \{1,\dots,n-1\}$ is

one-to-one and onto

Apply induction to get $k-1 = n-1 \Rightarrow k = n$.

<u>Th</u>: $f : \{1,\dots,n\} \to X$ both one-to-one and onto

$g : \{1,\dots,k\} \to X$ then $k = n$

<u>proof</u> $g^{-1} \circ f : \{1,\dots,n\} \to \{1,\dots,k\}$ is one-to-one and onto. then $n = k$.

one and onto. Then $n = k$.

<u>Def</u>: If $X = \emptyset$, we say $\# \emptyset = 0$. We still say the X is finite.

<u>Obs</u>: $\{1, 2, \dots, n\}$ is finite and $\#\{1, \dots, n\} = n$

<u>proof</u>: Let $i : \{1, \dots, n\} \to X = \{1, \dots, n\}$ $\quad i(\ell) = \ell$ for all $\ell \in \{1, \dots, n\}$. i is one to one and onto, then $\{1, \dots, n\}$ is finite and $\#\{1, \dots, n\} = n$.

<u>Obs</u>: $X' \subset X$ and X finite. Then X' is also finite and $\# X' \leq \# X$

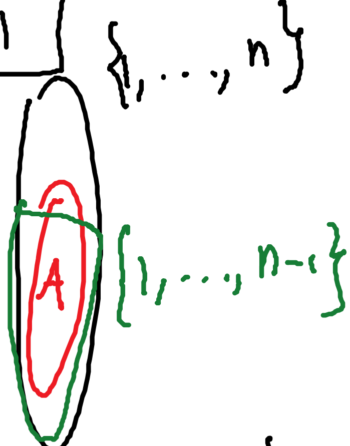
<u>Lemma</u>: $A \subset \{1, \dots, n\} \Rightarrow A$ is finite and $\# A \leq n$.

<u>proof</u>: By induction on n

$\boxed{n = 1} \Longrightarrow A = \{1\}$ finite and $\# A = 1$

or $A = \emptyset$ finite $\# A = 0$ ✓

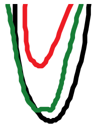
$\boxed{n > 1}$



$A' = A \cap \{1, \dots, n-1\}$

$A' \subset \{1, \dots, n-1\}$

Use induction to conclude that $\# A' \leq n-1$. If $A' \neq \emptyset$, $\exists k$



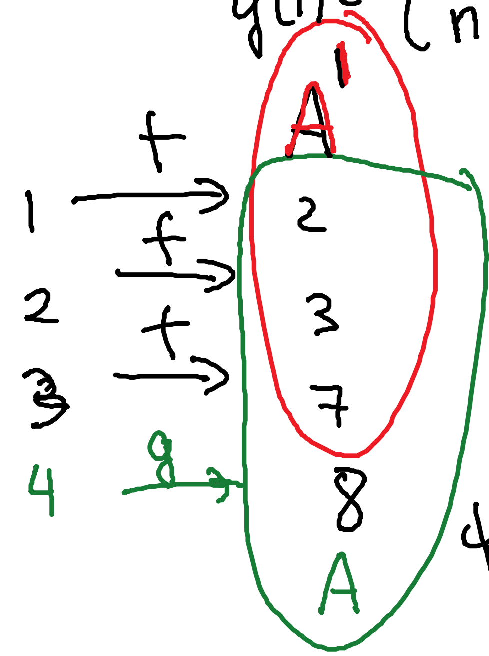
Conclude that $\# A' \le n-1$. If $A' \ne \emptyset$, $\exists k$ such that $k \le n-1$ and some function $f:\{1,\dots,k\} \to A'$ that is one-to-one and onto

If $n \notin A \Rightarrow A' = A \Rightarrow \#(A) = k \le n-1$ ✓

If $n \in A$ $\quad g:\{1,\dots,k+1\} \to A$

$$g(i) = \begin{cases} f(i) & \text{if } 1 \le i \le k \\ n & \text{if } i = k+1 \end{cases}$$

$n = 8$



g is onto one and onto.

This implies that $\#(A) = \underbrace{k}_{\le n-1} + 1 \le (n-1)+1 \le n$ ✓

Case $A' = \emptyset \Rightarrow A = \{n\}$ or $A = \emptyset$

<u>Def</u>: X is said to be infinite if it is not finite. In this case, we

write $\# X = \infty$

Lemma: If $\# X = \infty \Rightarrow \exists f: \mathbb{N} \to X$ that is one-to-one

proof: $\# X = \infty \Rightarrow X \neq \emptyset$

Let $x_1 \in X$. $f(1) = x_1$. Let $n > 1$.

Assume we have $f(1) = x_1, \ldots, f(n-1) = x_{n-1}$ with $x_i \neq x_j$ if $i \neq j$ $1 \leq i, j \leq n-1$ and $x_i \in X$.

Let $X^{(n)} = X - \{x_1, \ldots, x_{n-1}\}$.

If $X^{(n)} = \emptyset \Rightarrow X = \{x_1, \ldots, x_{n-1}\} \Rightarrow$ $\#(X) = n-1$ became $f: \{1, \ldots, n-1\} \to X$ would be one to one and onto. Contradiction. then $X^{(n)} \neq \emptyset$. thus, select $x_n \in X^{(n)}$ and set $f(n) = x_n$. Since $x_n \in X^{(n)} = X - \{x_1, \ldots, x_{n-1}\}$ then $x_n \neq x_i$ for all $1 \leq i < n$

thus, $f: \mathbb{N} \to X$ is one to one.

Prop: If $f: \mathbb{N} \to X$ is one-to-one, then

$\# X = \infty$

proof: If X is finite. Let $n = \# X$.

Let $X' = f(\{1, \dots, n+1\})$.

$f|_{\{1, \dots, n+1\}} : \{1, \dots, n+1\} \longrightarrow X'$ is one to one and onto then $X' \subset X$ but

$\# X' > \# X$ impossible