$$d((x,y),(x',y')) = \begin{cases} |y|+|y'|+|x-x'| & \text{if } x \neq x' \\ |y-y'| & \text{if } x = x' \end{cases}$$

1) If $x \neq x' \Rightarrow d((x,y),(x',y')) = |y|+|y'|+$
$|x-x'| \geq |x-x'| > 0$.

If $x = x \Rightarrow d((x,y),(x',y')) = |y-y'| \geq 0$

then $d((x,y),(x',y')) \geq 0 \quad \forall (x,y),(x',y') \in \mathbb{R}^2$.

If $x \neq x'$ $d((x,y),(x',y')) > 0$. Then

$d((x,y),(x',y')) = 0 \iff x = x'$ & $d((x,y),(x',y')) = |y-y'| = 0$

$\iff x = x'$ & $y = y' \iff (x,y) = (x',y')$.

3) $(x,y),(x',y'),(x'',y'') \in \mathbb{R}^2$.

We need to show $d((x,y),(x'',y'')) \leq d((x,y),(x',y')) +$

$+ d((x',y'),(x'',y''))$

Case I $x = x' = x''$. Case II $x = x''$ but $x \neq x'$

Case III $x \neq x''$

Case I $d((x,y),(x'',y'')) = |y-y''| \le |y-y'| + |y'-y''|$

$= d((x,y),(x',y')) + d((x',y'),(x'',y''))$

Case II $d((x,y),(x'',y'')) = |y-y''|$

$d((x,y),(x',y')) = |y| + |y'| + |x-x'|$

$d((x',y'),(x'',y'')) = |y'| + |y''| + |x'-x''|$

$d((x,y),(x'',y'')) = |y-y''| \le |y| + |y''| \le |y| + |y'| + |x-x'| + |y''| + |y'| + |x'-x''| = d((x',y'),(x,y)) + d((x',y'),(x'',y''))$ ✓

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12 Ch 3

$$x_{n+1} = x_n + \frac{1}{x_n^2} \qquad x_1 = 1$$

$S = \{x_1, \ldots, x_n, \ldots\}$ If S is bounded from above,

Let $b = \text{lub}(S) > 0$. Let $\varepsilon > 0$. $b - \varepsilon$ is not
$b - \frac{1}{2b^2}$
an upper bound because $b - \varepsilon < b = \text{lub}(S)$. Thus, $\exists$
n such that $b - \varepsilon < x_n \le b$

$x_{n+1} = x_n + \frac{1}{x_n^2} > b - \varepsilon + \frac{1}{b^2}$ Let $\varepsilon = \frac{1}{2b^2} > 0$

$x_{n+1} > b + \frac{1}{2b^2} > b$ contradiction.

$x_{n+1} > b$

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5) $U \subseteq \mathbb{R}$ U open bounded. Show that $\exists\, I_\alpha = (a_\alpha, b_\alpha)$

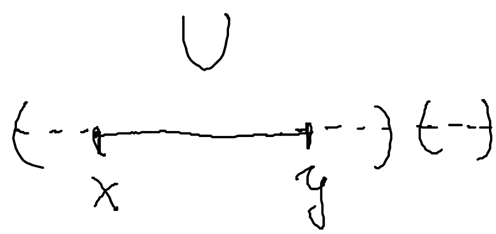
$\alpha \in \Lambda$ such that $U = \bigcup_{\alpha \in \Lambda} I_\alpha$ & $I_\alpha \cap I_\beta = \emptyset$

if $\alpha \neq \beta$

~~Let x, y ∈ U. We say x ~ y if ∃~~

Let $x \in U$.

$I_x = \{ y \in U : [x,y] \subset U \text{ or } [y,x] \}$



$U = \bigcup_{x \in U} I_x$

I want to show that each I_x is open.

I want to show that $I_x \cap I_y \neq \emptyset \Rightarrow$

$I_x = I_y$

~~S = { select exactly one x~~

$U = \bigcup I_x$

$$U = \bigcup I_x$$

~~$x \in S$~~ not repeated

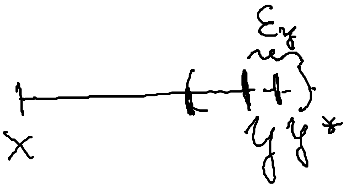
~~$U \in S$~~

if $y \ge x$ then

1) I_x is open. Let $y \in I_x$ $[x,y] \subset U$

Assume $y > x$. Since U is open, $\exists\, \varepsilon_y > 0$ such that $B_{\varepsilon_y}(y) \subset U$. Let $y^* \in B_{\varepsilon_y}$, $0 < \varepsilon_y \le y - x$. Then, $[x, y^*] \subset U$

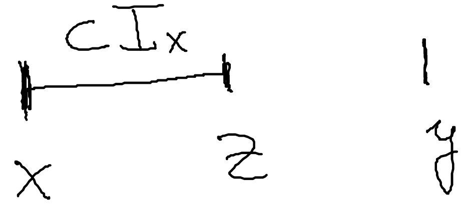
~~Let $y^* \in$~~ then $y^* \in I_x$.



Then $B_{\varepsilon_y}(y) \subset I_x \Rightarrow I_x$ is open.

2) $I_x \cap I_y \ne \emptyset$. Let $z \in I_x \cap I_y$. Assume

$x \le z \le y$

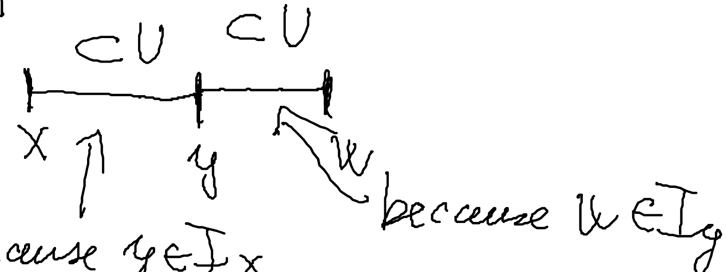


$[x,z] \subset U$ because $z \in I_x$

$[z,y] \subset U$ because $z \in I_y$

Then $[x,y] = \underbrace{[x,z]}_{\subset U} \cup \underbrace{[z,y]}_{\subset U} \subset U \Rightarrow y \in I_x$

Let $w \in I_y$. Then



then $[x, w] \subset U \Rightarrow w \in I_x$. then $I_y \subset I_x$

Similarly $I_x \subset I_y$. $I_x = I_y$ if $I_x \cap I_y \neq \emptyset$