Exam 2 - MATH 4317

The exam is to be done individually without the help of anyone. You are not to talk about the exam with anyone.

Last name:

First name:

Problem 1 (5 points): Let d and d' be two distances on the same set E such that $d(x,y) \leq d'(x,y)$ for all $x, y \in E$

1) Let x_n be a sequence in E that converges with the distance d'. Prove that x_n also converges with the distance d.

2) Show that if U is open with the distance d, then U is open with the distance d'.

3) Show an example of E, d, d' and x_n that x_n converges with the distance d but not with the distance d'.

Problem 2 (5 points): Let $f : \mathbb{R} \to \mathbb{R}$. Assume f is increasing, i.e. x < y implies f(x) < f(y). Let $x_0 \in \mathbb{R}$. Assume $f(x_0)$ is a cluster point of $f(\mathbb{R}) \cap [f(x_0), \infty)$ and $f(x_0)$ is also a cluster point of $f(\mathbb{R}) \cap (-\infty, f(x_0)]$. Prove that f is continuous at x_0 .

Problem 3 (5 points): Let

$$S = \{0\} \cup \left\{\frac{1}{n} : n \text{ positive integer}\right\}.$$

Is S compact or not? Prove it.

Problem 4 (5 points): Let *E* be a metric space. Let $S \subset E$. Assume *S* is compact. Let $f: E \to \mathbb{R}$ be defined as $f(x) = \text{g.l.b.}\{d(x, s) : s \in S\}$.

- 1) Prove that, for all $x \in E$ there exists $u \in S$ such that f(x) = d(x, u).
- 2) Prove that f is continuous.