Exam 1 - MATH 4317

Show all your work.

Section AU or AG?:

Last name:

First name:

list the 3 problems you want graded:

Problem 1 (5 points): Show that the subset of $\mathbf{R}^2$ given by $\{(x,y) \in \mathbf{R}^2 : x > y\}$ is open.

Let $(x_0, y_0) \in S = \{(x,y) : x > y\}$

~~Let~~ If $d((x,y),(x_0,y_0)) < \varepsilon$ then

$x > x_0 - \varepsilon$ and $y < y_0 + \varepsilon$ (because

$|x - x_0| \le d((x,y),(x_0,y_0))$ & $|y - y_0| \le d((x,y),(x_0,y_0))$).

then $x > x_0 - \varepsilon = x_0 - y_0 + y_0 - \varepsilon = (x_0 - y_0) + \underbrace{(y_0 + \varepsilon)}_{> y}$

$- 2\varepsilon > (x_0 - y_0) - 2\varepsilon + y$

select $\varepsilon = \frac{x_0 - y_0}{2} > 0$. then $x > y$. then

$B_{\left(\frac{x_0 - y_0}{2}\right)}(x_0, y_0) \subset S \Rightarrow S$ is open

Problem 2 (5 points): Let S be a subset of the metric space E. A point $p \in S$ is called an interior point of S if there is an open ball in E of center p which is contained in S. Prove that the set of interior points of S is an open subset of E (called the interior of S) that contains all other open subsets of E that are contained in S.

Claim 1: U open, $U \subset S \Rightarrow U \subset \mathrm{int}(S)$ (interior of S)

proof: $x \in U \Rightarrow \exists \varepsilon > 0 : B_\varepsilon(x) \subset U$ (because U is open). ~~But~~ then $B_\varepsilon(x) \subset U \subset S$. then $x \in \mathrm{int}(S)$

then $U \subset \mathrm{int}(S)$

Claim 2: $x \in \mathrm{int}(S) \Rightarrow \exists\, U$ open such that $x \in U \subset S$.

proof: $x \in \mathrm{int}(S) \Rightarrow \exists\, \varepsilon > 0$ such that $B_\varepsilon(x) \subset S$

$U = B_\varepsilon(x)$ works.

claim 3: $\mathrm{int}(S) = \bigcup_{\substack{U \text{ open} \\ U \subset S}} U$

proof: Claim 1 implies $\bigcup_{\substack{U \text{ open} \\ U \subset S}} U \subset \mathrm{int}(S)$

Claim 2 implies $\mathrm{int}(S) \subset \bigcup_{\substack{U \text{ open} \\ U \subset S}} U$

claim 4: int(S) is open (from claim 3 as ~~this~~ int(S) is the union of open sets) Claim 4 & Claim 1 complete the proof

Problem 3 (5 points): Prove that a subset of a metric space is closed if and only if it contains all its cluster points.

$\Rightarrow$) Let S be closed. Let $x \notin S$. ~~then~~ Since S is closed, $\exists\, \varepsilon > 0$ such that $B_\varepsilon(x) \cap S = \emptyset$ then x is not a cluster point of S. ~~then~~ then S contains all its cluster points

$\Leftarrow$) Assume S contains all its cluster points. Let $x \notin S$. Then x is not a cluster point of S. Then $\exists\, \varepsilon > 0$ such that $B_\varepsilon(x) \cap S = \text{finite set} = \{x_1, \dots, x_n\}$.

$x_i \neq x$ $\forall i$ because $x \notin S$. Let $\delta = \min_{1 \le i \le n} \{d(x, x_i)\}$

then $B_\delta(x) \cap S = \emptyset$ & $\delta > 0 \Rightarrow$

S is closed.

Problem 4 (5 points): Let E be a compact space. U_i, $i \in \mathbf{N}$ be an infinite family of non-empty open sets such that $U_i \cap U_j = \emptyset$ if $i \neq j$. Show that this family of open sets does not cover E.

Let $V_n = \bigcup_{i=1}^{n} U_i$. Then $V_{n+1} \supset V_n$ and each V_n is open. Then $V_n^c \supset V_{n+1}^c$ and each V_n^c is closed. $U_{n+1} \cap V_n = \phi$ & $U_{n+1} \neq \phi \Rightarrow V_n \neq E$. Thus $V_n^c \neq \phi$.

Since E is compact & $V_{n+1}^c \subset V_n^c$ and V_n^c is closed & non-empty $\forall n$, $\exists x \in E$ such that

$$x \in \bigcap_{n=1}^{\infty} V_n^c \Rightarrow x \notin \bigcup_{n=1}^{\infty} V_n \supset \bigcup_{n=1}^{\infty} U_n$$

the family U_n does not cover E.