Homework 2 - MATH 4317

Five problems from this list will be in the final.

Problem 1 (5 points): Assume that a metric space E has the property that the intersection of any collection of open sets is open. Which subsets of E are open?

Let $x \in E$.

Claim 1 $\{x\} = \bigcap_{r>0} B_r(x)$

pf: $\subset$) $x \in B_r(x)\ \forall r>0 \Rightarrow x \in \bigcap_{r>0} B_r(x)$

$\Rightarrow \{x\} \subset \bigcap_{r>0} B_r(x)$

$\supset$) If $y \neq x \Rightarrow r_0 = d(x,y) > 0 \Rightarrow y \notin B_{r_0}(x)$

$\Rightarrow y \notin \bigcap_{r>0} B_r(x)$ ∎

Since $B_r(x)$ is open $\forall r>0$ and $\bigcap$ of open is open ~~for~~ in this problem, $\{x\}$ is open.

Let $S \subset E$. $S = \bigcup_{x \in S} \{x\}$ which is a union of opens, and thus S is open.

Thus, any subset of E is open

Prob 2

Claim 1 f is one-to-one

Claim 2 f^{-1} exists

Claim 3 f^{-1} is increasing (strictly)

Claim 4 f is cont

Claim 5 f^{-1} is cont

$\boxed{\text{pf of claim 1}}$ If $x < y \Rightarrow f(x) < f(y)$ (bc f strict inc)

If $y < x \Rightarrow f(y) < f(x)$ (because f strict increasing).

thus, if $x \neq y \Rightarrow x < y$ or $y < x \Rightarrow f(x) \neq f(y)$ in either case

$\boxed{\text{pf claim 2}}$ f onto by hypothesis. one to one by claim 1 $\Rightarrow f$ onto one-to-one $\Rightarrow f$ has an inverse.

$\boxed{\text{pf claim 3}}$ $y_1 < y_2$. $y_1, y_2 \in V$. $x_1 = f^{-1}(y_1)$

$x_2 = f^{-1}(y_2)$. If $x_2 \leq x_1 \Rightarrow f(x_2) = y_2 \leq f(x_1) = y_1$

impossible $\Rightarrow x_1 < x_2 \Rightarrow f$ strictly increasing.

$\boxed{\text{pf claim 4}}$ $x_0 \in U$. $y_0 = f(x_0)$. $\varepsilon > 0$.

$\varepsilon' > 0$ such that $B_{\varepsilon'}(y_0) \subset V$ & $\varepsilon' \leq \varepsilon$.

$y_0 - \varepsilon' < y_0 < y_0 + \varepsilon' \Rightarrow x_\ell = f^{-1}(y_0 - \varepsilon') < x_0 =$

$= f^{-1}(y_0) < x_r = f^{-1}(x_0+\varepsilon')$. Let $\delta = \min\{x_0 - x_\ell, x_r - x_0\}$. If $|x - x_0| < \delta \Rightarrow$

$$x_\ell \le x_0 - \delta < x < x_0 + \delta \le x_r \Rightarrow$$

$$f(x_\ell) < f(x) < f(x_r) = y_0 + \varepsilon' \le y_0 + \varepsilon$$

$$y_0 - \varepsilon \le y_0 - \varepsilon' = f(x_\ell) < f(x)$$ then

$$|f(x) - \underbrace{y_0}_{= f(x_0)}| < \varepsilon$$

$\boxed{\text{pf Claim 5}}$ same as claim 4, change roles of f & f^{-1}

Problem 3 (5 points): Let E be metric space with more than one point. Let $x_0 \in E$. Assume $\{x_0\}$ is open. Is E connected? Prove it.

No. $\{x_0\}$ is always closed

pf: $y \notin \{x_0\} \Rightarrow y \neq x_0 \Rightarrow$
$r = d(x_0, y) > 0 \Rightarrow$ if $d(x, y) < r$,
$d(x_0, x) + d(x, y) \geq d(x_0, y)$
$d(x_0, x) \geq r - d(x, y) > 0$
$\Rightarrow x \neq x_0 \Rightarrow B_r(y) \subset \{x_0\}^c \Rightarrow$
$\{x_0\}^c$ is closed $\Rightarrow \{x_0\}$ is closed.

$\{x_0\}$ in this ~~open~~ space is both closed & open. $\{x_0\} \neq \emptyset$. $\{x_0\} \neq E$ because E have more than 1 point. $\Rightarrow$ E is not connected

$C_\varepsilon(x) = \{y : d(x,y) \le \varepsilon\}$

Problem 4 (5 points): Assume that a metric space E is complete and has the following property:

If x_n is a sequence and there exists $\varepsilon > 0$ such that $d(x_i, x_j) > \varepsilon$ for all $i \neq j$, then the sequence x_n is not bounded.

Prove that if $S \subset E$ is bounded and closed, the S is compact. (Hint. You can use this fact: If every sequence has a convergent subsequence, then E is compact)

$\boxed{\text{Claim 1}}$: $\forall \varepsilon > 0$ ~~$\exists$~~ $\exists$ n positive integer and $s_1, \dots, s_n \in S$ such that $S \subset \bigcup_{i=1}^{n} B_\varepsilon(s_i)$

pf: ~~Effector~~ Let $\varepsilon > 0$. Let $s_1 \in S$.

Select ~~s_2~~ $s_2, s_3, \dots$ inductively as follows.

If $S \subset \bigcup_{i=1}^{n} B_\varepsilon(s_i)$, done. Otherwise,

select $s_{n+1} \in S - \bigcup_{i=1}^{n} B_\varepsilon(s_i)$. Note that

$d(s_{n+1}, s_i) \geq \varepsilon$ $\forall 1 \le i \le n$. Thus,

$d(s_i, s_j) \geq \varepsilon$ $\forall i \neq j$ in this sequence. Since

S is bounded, $\exists$ n such that $S \subset \bigcup_{i=1}^{n} B_\varepsilon(s_i)$,

otherwise $\{s_1, s_2, \dots\} \subset S$ would be unbounded

by the hypothesis.

Let U_α open in E, $\alpha \in I$ such that $S \subset \bigcup_{\alpha \in I} U_\alpha$. Assume there is not a finite subcover.

Let $\varepsilon > 0$ $S \subset \bigcup_{i=1}^{n} C_\varepsilon(s_i)$ for some n & $s_1, \dots, s_n$ $\Rightarrow \bigcup_{i=1}^{n} \left(S \cap C_\varepsilon(s_i) \right) \subset \bigcup_{\alpha \in I} U_\alpha \Rightarrow \exists\, i$ such that $S \cap C_\varepsilon(s_i)$ can not be covered by a finite number of the U_α.

With some work but same ideas, we can prove $\exists S \supset S_1 \supset S_2 \supset \dots$ such that

$S_1 = S \cap C_1(s_1)$ for some $s_1 \in S$

$S_{n+1} = S_n \cap C_{\frac{1}{n+1}}(s_{n+1})$ for some $s_{n+1} \in S_n$

S_i can not be covered by a finite number of the U_α.

S_n is cauchy. $S_n \to s$. $s \in S \Rightarrow$

$\exists\, x_0$ such that $s \in U_{x_0} \Rightarrow \exists\, \varepsilon > 0$

such that $B_\varepsilon(s) \subset U_{x_0}$ and it can

be proved that $S_N \subset B_\varepsilon(s) \subset U_{x_0}$

if $N > \frac{2}{\varepsilon}$. Contradiction