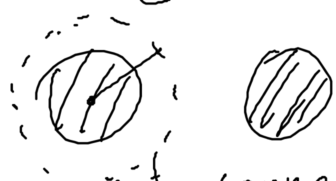
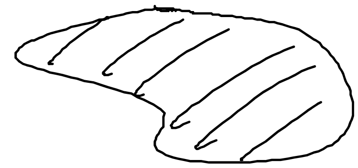
Conectedness

E



not connected



connected

Def: E is connected if E & ϕ are the only sets that are both open and closed. (in E)

Obs: E is not connected $\iff \exists\, A, B$ open sets such that $A \cap B = \phi$, $A \neq \phi$, $B \neq \phi$ & $A \cup B = E$.

$\Rightarrow$)

$\phi \subsetneq A \subsetneq E$ $\qquad$ A open & closed

$$E = \underset{\text{open}}{A} \cup \underset{\text{open}}{A^c} \qquad A^c = B$$

$\Leftarrow$) $E = A \cup B$ A, B open, then A closed because $A = B^c$ $(A \cap B = \phi)$.

Ex: 1) $E = \{x_0, x_1\}$ $\qquad d(x_0, x_1) = r > 0$.

$\{x_0\} = B_\varepsilon(x_0) \qquad 0 < \varepsilon < r$

$E = \{x_0\} \amalg \{x_1\}$ not connected

2) $\underbrace{[0,1)}_{\text{open}} \cup \underbrace{(1,2]}_{\text{open}} = E$ not connected

3) $\mathbb{N}$ not connected $\quad S = \{x_1, \dots, x_r, \dots\} \subset \mathbb{N}$

$x_i \not< x_{i+1}$

$$S = \bigcup_{x \in S} B_1(x)$$

4) $\mathbb{Q}$ is not connected

$$\mathbb{Q} = \underbrace{\{x \in \mathbb{Q} : x < \sqrt{2}\}}_{\text{open}} \cup \underbrace{\{x \in \mathbb{Q} : x > \sqrt{2}\}}_{\text{open}}$$

<u>Obs</u>: E metric space. $S \subset E$. Then S is also a metric space with the same distance.

1) Let $x_0 \in S$. Let $\varepsilon > 0$.

ball centered at x_0 with radius ε in S = ball centered at x_0 with radius ε in E $\cap S$

2) If $U \subset E$ & U is open in E $\Rightarrow$ $U \cap S$ is open in S

<u>Ex</u> $(-\infty, \sqrt{2})$ is open in $\mathbb{R}$ $\Rightarrow$ $(-\infty, \sqrt{2}) \cap \mathbb{Q}$ is open in $\mathbb{Q}$

<u>proof of 3</u> Let $x \in U$. $\exists\, \varepsilon_x > 0$ such that $B^E_{\varepsilon_x}(x) = \{y \in E : d(x,y) < \varepsilon_x\} \subset U$. Then

$$U = \bigcup_{x \in U} B^E_{\varepsilon_x}(x)$$

$$\Rightarrow U \cap S = \bigcup (B^E_{\varepsilon_x}(x) \cap S) \quad \text{open in } S$$

then $U \cap S = \bigcup_{x \in U} \underbrace{(B^E_{\varepsilon_x}(x) \cap S)}_{B^S_{\varepsilon_x}(x) = \{y \in S : d(x,y) < \varepsilon_x\}}$ open in S

3) $V \subset S$, V open in S $\Rightarrow$ $\exists\, U \subset E$ such that U is open & $V = U \cap S$

~~proof~~: $V = \bigcup_{x \in V} B^S_{\varepsilon_x}(x) = \bigcup_{x \in V} (B^E_{\varepsilon_x}(x) \cap S) =$

$$= \underbrace{\left(\bigcup_{x \in V} B^E_{\varepsilon_x}(x)\right)}_{= U} \cap S$$

Th: E metric space. $S \subset E$. S is not connected $\Leftrightarrow$ $\exists\, A, B$ both open in E such that $S \subset A \amalg B$

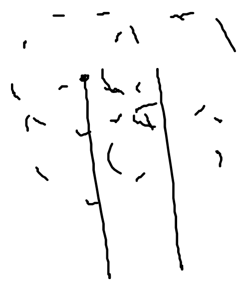
$A \cap S \neq \emptyset$, $B \cap S \neq \emptyset$

disjoint union means $A \cap B = \emptyset$ (regular union +)

~~proof~~: $\Leftarrow$) $S = \underset{\substack{\text{open in } S \\ \neq \emptyset}}{(A \cap S)} \amalg \underset{\substack{\text{open in } S \\ \neq \emptyset}}{(B \cap S)}$

$\Rightarrow$) $S = V_1 \amalg V_2$ $\quad$ V_1 open in S $\quad V_1 \neq \emptyset$
V_2 open in S $\quad V_2 \neq \emptyset$

Let $x \in V_1$ $\exists\, \varepsilon_x > 0$ such that



Let $x \in V_1$ $\exists\ \varepsilon_x > 0$ such that

$B^S_{\varepsilon_x}(x) \subset V_1$

Let $x \in V_2$ $\exists\ \varepsilon_x > 0$ such that

$B^S_{\varepsilon_x}(x) \subset V_2$

$$U_1 = \bigcup_{x \in V_1} B^E_{\frac{\varepsilon_x}{2}}(x) \qquad U_2 = \bigcup_{x \in V_2} B^E_{\frac{\varepsilon_x}{2}}(x)$$

Claim 1) U_1 & U_2 open in E.

Claim 2) $U_1 \cap S = V_1$ $\quad U_2 \cap S = V_2$

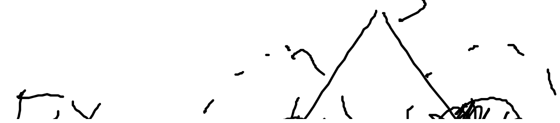
Claim 3) Let $x \in V_1$ & $y \in V_2 \Rightarrow B^E_{\frac{\varepsilon_x}{2}}(x) \cap B^E_{\frac{\varepsilon_y}{2}}(\cancel{x}y) = \phi$

Assume it is not. Let $z \in B^E_{\frac{\varepsilon_x}{2}}(x) \cap B^E_{\frac{\varepsilon_y}{2}}(y)$. then

$$d(x,y) \le \underbrace{d(x,z)}_{< \frac{\varepsilon_x}{2}} + \underbrace{d(z,y)}_{< \frac{\varepsilon_y}{2}} < \frac{\varepsilon_x}{2} + \frac{\varepsilon_y}{2}$$

If $\varepsilon_x \ge \varepsilon_y \Rightarrow d(x,y) < \varepsilon_x$ impossible, because $y \in V_2$ and $B^S_{\varepsilon_x}(x) \cap V_2 = \phi$

Ex: $[0,1) \cup (1,2] \subset \underset{\text{open}}{(-\infty,1)} \amalg \underset{\text{open}}{(1,\infty)}$



Ex