Obs 1: Let $M>0$. $\forall x \in \mathbb{R}^n$ $\exists a \in \mathbb{Z}^n$ such that $d(x, \frac{a}{M}) < \frac{\sqrt{n}}{M}$

proof: Let a_i such that $\frac{a_i}{M} \le x_i < \frac{a_i}{M} + \frac{1}{M}$

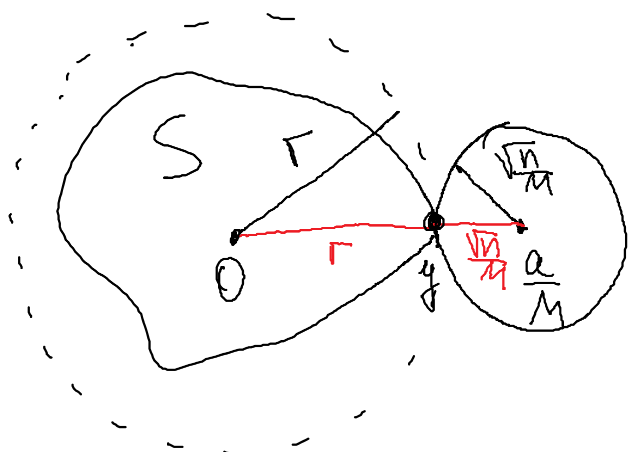
$$d\left(x, \frac{a}{M}\right) = \sqrt{\sum_{i=1}^{n} \left(x_i - \frac{a_i}{M}\right)^2} < \sqrt{\sum_{i=1}^{n} \left(\frac{1}{M}\right)^2} = \frac{\sqrt{n}}{M}$$

Obs 2: Let $S \subset \mathbb{R}^n$. Let $r>0$ such that $S \subset B_r(0)$

If $C_{\frac{\sqrt{n}}{M}}\left(\frac{a}{M}\right) \cap S \neq \emptyset$, with $a \in \mathbb{Z}^n$, $M>0$,

then $|a_i| \le \sqrt{n} + rM$

proof: Recall $C_\varepsilon(x) = \{ y : d(x,y) \le \varepsilon \}$



Let $y \in C_{\frac{\sqrt{n}}{M}}\left(\frac{a}{M}\right) \cap S$

then

$$d\left(\frac{a}{M}, 0\right) \le \underbrace{d(0,y)}_{<r} + \underbrace{d\left(y, \frac{a}{M}\right)}_{\le \frac{\sqrt{n}}{M}}$$

$$\frac{|a_i|}{M} \le d\left(\frac{a}{M}, 0\right) < r + \frac{\sqrt{n}}{M} \Rightarrow |a_i| < \sqrt{n} + Mr$$

for any i

Obs 3: Let $S \subseteq \mathbb{R}^n$. S bounded. Let $\varepsilon > 0$. Assume $S \subset B_r(0)$. Then $S \subset \bigcup_{\substack{a \in \mathbb{Z}^n \\ |a_i| \le \sqrt{n}(1+\frac{r}{\varepsilon})}} C_\varepsilon\left(\frac{a}{\frac{\sqrt{n}}{\varepsilon}}\right)$

proof: Use Obs 1 with $M = \frac{\sqrt{n}}{\varepsilon}$. $\forall x \in \mathbb{R}^n$ $\exists$ $a \in \mathbb{Z}^n$ such that $d\left(x, \frac{a}{\frac{\sqrt{n}}{\varepsilon}}\right) < \frac{\sqrt{n}}{\frac{\sqrt{n}}{\varepsilon}} = \varepsilon$

$$\mathbb{R}^n \subset \bigcup_{a \in \mathbb{Z}^n} C_\varepsilon\left(\frac{a}{\frac{\sqrt{n}}{\varepsilon}}\right)$$. Intersect with S

$$S \subset \bigcup_{a \in \mathbb{Z}^n} \left(C_\varepsilon\left(\frac{a}{\frac{\sqrt{n}}{\varepsilon}}\right) \cap S\right) \quad (*)$$

Use Obs 2 with $M = \frac{\sqrt{n}}{\varepsilon}$. It say that if $C_\varepsilon\left(\frac{a}{\frac{\sqrt{n}}{\varepsilon}}\right) \cap S \neq \emptyset$ then $|a_i| \le \sqrt{n} + r\frac{\sqrt{n}}{\varepsilon} = \sqrt{n}\left(1 + \frac{r}{\varepsilon}\right)$. Then from $(*)$ we get

$$S \subset \bigcup_{\substack{a \in \mathbb{Z}^n \\ |a_i| \le \sqrt{n}(1+\frac{r}{\varepsilon})}} C_\varepsilon\left(\frac{a}{\frac{\sqrt{n}}{\varepsilon}}\right)$$

Obs 4: $S \subset \mathbb{R}^n$. S bounded. $\varepsilon > 0$. Then $\exists$ $x_1, x_2, \dots, x_l \in \mathbb{R}^n$ such that $S \subset \bigcup_{i=1}^{l} C_\varepsilon(x_i)$

$\exists\ x_1, x_2, \dots, x_\ell \in \mathbb{R}^n$ such that $S \subset \bigcup_{i=1}^{\ell} C_\varepsilon(x_i)$

<u>proof</u>: From Obs 3, let r such that $S \subset B_r(0)$

then $S \subset \bigcup_{\substack{a \in \mathbb{Z}^n \\ |a_i| \le \sqrt{n}\left(1+\frac{r}{\varepsilon}\right)}} C_\varepsilon\left(\frac{a}{\frac{\sqrt{n}}{\varepsilon}}\right)$ this is a union of at most $\left(2\sqrt{n}\left(1+\frac{r}{\varepsilon}\right)+1\right)^n$

<u>Theorem</u>: $S \subset \mathbb{R}^n$. S closed & bounded $\Rightarrow$ S compact

<u>proof</u>: Let $S \subset \bigcup_{i \in I} V_i$ V_i open.

Assume S can not be covered with a finite subfamily of the V_i.

Let $\varepsilon_1 = 1$. From Obs 4, $S = \bigcup_{i=1}^{N_1} \underbrace{C_1(x_i) \cap S}_{\text{closed}} \subset \bigcup_{i \in I} V_i$

for some N_1 integer & $x_1, \dots, x_{N_1}$

Then, $\exists\, i_1$ such that $S_1 = C_1(x_{i_1}) \cap S$ can not be covered by a finite subfamily of the V_is.

Do the same with S_1 instead of S and $\varepsilon_2 = \frac{1}{2}$ instead of $\varepsilon_1 = 1$. We get $\exists\ S_2 = C_{1/2}(y_2) \cap S_1$ that can not be covered by a finite subfamily of V_is

Go on

$S \supset S_1 \supset S_2 \supset S_3 \cdots \supset S_n \supset S_{n+1} \supset \cdots$

Each S_i is closed. $S_i \subset C_{1/i}(y_i)$ for some y_i and $S_i \neq \emptyset$ and S_i can not be covered by a finite number of V_i s.

Let $x_i \in S_i$. Let $\varepsilon > 0$. Let $N > \frac{2}{\varepsilon}$.

If $n, m \geq N$ then $d(x_n, x_m) \leq \underbrace{d(x_n, y_N)}_{\leq \frac{1}{N}} + \underbrace{d(y_N, x_m)}_{\leq \frac{1}{N}} \leq \frac{2}{N} < \varepsilon$

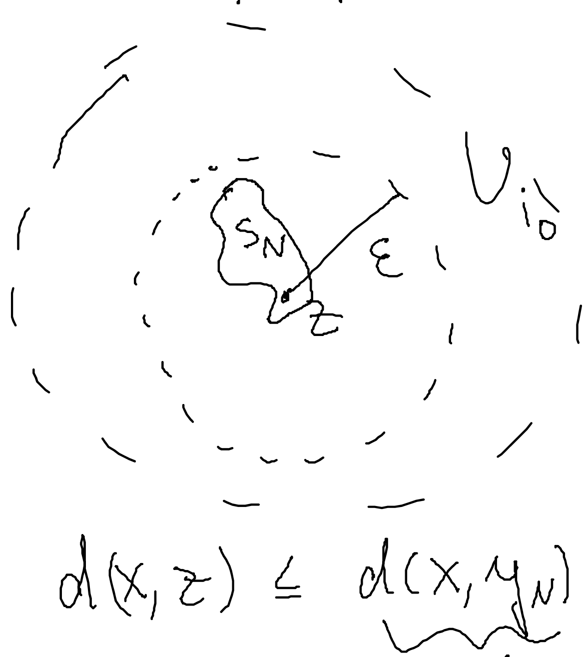
$x_n \in S_n \subset S_N \subset C_{1/N}(y_N)$

$x_m \in S_m \subset S_N \subset C_{1/N}(y_N)$

Then $x_n \longrightarrow z$ for z because $\mathbb{R}^n$ is complete.

S_n is closed & $x_m \in S_n$ if $m \geq n$ then $z \in S_n$ for all n. Note $S_n \subset S$, thus $z \in S$

thus, $z \in V_{i_0}$ for some $i_0 \in I$.



Let $\varepsilon > 0$ such that $B_\varepsilon(z) \subset V_{i_0}$

Let $N > \frac{2}{\varepsilon}$. Then, if $x \in S_N \subset C_{\frac{1}{N}}(y_N)$, then

$$d(x, z) \leq \underbrace{d(x, y_N)}_{\leq \frac{1}{N}} + \underbrace{d(y_N, z)}_{\leq \frac{1}{N}} \leq \frac{2}{N} < \varepsilon$$

because both z & $x \in S_N$.

z & $x \in S_N$

Thus $S_N \subset B_\varepsilon(z) \subset U_{i_0}$.

This is a contradiction, because we assumed none of the S_n could be covered by a finite subfamily of the U_i s.