CHAPTER II

The Real Number System

The real numbers are basic to analysis, so we must have a clear idea of what they are. It is possible to construct the real number system in an entirely rigorous manner, starting from careful statements of a few of the basic principles of set theory,* but we do not follow this approach here for two reasons. One is that the detailed construction of the real numbers, while not very difficult, is time-consuming and fits more properly into a course on the foundations of arithmetic, and the other reason is that we already "know" the real numbers and would like to get down to business. On the other hand we have to be sure of what we are doing. Our procedure in this book is therefore to *assume* certain basic properties (or axioms) of the real number system, all of which are in complete agreement with our intuition and all of which can be proved easily in the course of any rigorous construction of the system. We then sketch how most of the familiar properties of the real numbers are consequences of the basic properties assumed and how these properties actually completely determine the real numbers. The rest of the course will be built on this foundation.

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* The standard procedure for constructing the real numbers is as follows: One first uses basic set theory to define the natural numbers $\{1, 2, 3, \ldots\}$ (which, to begin with, are merely a set with an order relation), then one defines the addition and multiplication of natural numbers and shows that these operations satisfy the familiar rules of algebra. Using the natural numbers, one then defines the set of integers $\{0, \pm 1, \pm 2, \ldots\}$ and extends the operations of addition and multiplication to all the integers, again verifying the rules of algebra. From the integers one next obtains the rational numbers, or fractions. Finally, from the rational numbers one constructs the real numbers, the basic idea in this last step being that a real number is something that can be approximated arbitrarily closely by rational numbers. (The manufacture of the real numbers may be witnessed in E. Landau's *Foundations of Analysis.*)

§ 1. THE FIELD PROPERTIES.

We define the *real number system* to be a set $\mathbf{R}$ together with an ordered pair of functions from $\mathbf{R} \times \mathbf{R}$ into $\mathbf{R}$ that satisfy the seven properties listed in this and the succeeding two sections of this chapter. The elements of $\mathbf{R}$ are called *real numbers*, or just *numbers*. The two functions are called *addition* and *multiplication*, and they make correspond to an element $(a, b) \in \mathbf{R} \times \mathbf{R}$ specific elements of $\mathbf{R}$ that are denoted by $a + b$ and $a \cdot b$ respectively.

We speak of *the* real number system, rather than *a* real number system, because it will be shown at the end of this chapter that the listed properties completely determine the real numbers, in the sense that if we have two systems which satisfy our properties then the two underlying sets $\mathbf{R}$ can be put into a unique one-one correspondence in such a way that the functions $+$ and $\cdot$ agree. Thus the basic assumption made in this chapter is that *a* system of real numbers exists.

The five properties listed in this section are called the *field properties* because of the mathematical convention calling a *field* any set, together with two functions $+$ and $\cdot$, satisfying these properties. They express the fact that the real numbers are a field.

PROPERTY I. (COMMUTATIVITY). *For every* $a, b \in \mathbf{R}$, *we have* $a + b = b + a$ and $a \cdot b = b \cdot a$.

PROPERTY II. (ASSOCIATIVITY). For every $a, b, c \in \mathbf{R}$, *we have* $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

PROPERTY III. (DISTRIBUTIVITY). *For every* $a, b, c \in \mathbf{R}$, *we have* $a \cdot (b + c) = a \cdot b + a \cdot c$.

PROPERTY IV. (EXISTENCE OF NEUTRAL ELEMENTS). *There are distinct elements* 0 *and* 1 *of* $\mathbf{R}$ *such that for all* $a \in \mathbf{R}$ *we have* $a + 0 = a$ *and* $a \cdot 1 = a$.

PROPERTY V. (EXISTENCE OF ADDITIVE AND MULTIPLICATIVE INVERSES). *For any* $a \in \mathbf{R}$ *there is an element of* $\mathbf{R}$, *denoted* $-a$, *such that* $a + (-a) = 0$, *and for any nonzero* $a \in \mathbf{R}$ *there is an element of* $\mathbf{R}$, *denoted* a^{-1}, *such that* $a \cdot a^{-1} = 1$.

Most of the rules of elementary algebra can be justified by these five properties of the real number system. The main consequences of the field properties are given in paragraphs F 1 through F 10 immediately below, together with brief demonstrations. We shall employ the common notational conventions of elementary algebra when no confusion is possible.

For example, we often write ab for $a \cdot b$. One such convention is already implicit in the statement of the distributive property (Property III above), where the expression $a \cdot b + a \cdot c$ is meaningless unless we know the order in which the various operations are to be performed, that is how parentheses should be inserted; by $a \cdot b + a \cdot c$ we of course mean $(a \cdot b) + (a \cdot c)$.

F 1. In a sum or product of several real numbers parentheses can be omitted. That is, the way parentheses are inserted is immaterial. Thus if $a, b, c, d \in \mathbf{R}$, the expression $a + b + c + d$ may be defined to be the common value of $(a + (b + c)) + d = ((a + b) + c) + d = (a + b) + (c + d) = a + (b + (c + d)) = \cdots$; that these expressions with parentheses indeed possess a common value can be shown by repeated application of the associative property. The general fact (with perhaps more than four summands or factors) can be proved by starting with any meaningful expression involving elements of $\mathbf{R}$, parentheses, and several $+$'s or several $\cdot$'s, and repeatedly shoving as many parentheses as possible all the way to the left, always ending up with an expression of the type $((a + b) + c) + d$.

F 2. In a sum or product of several real numbers the order of the terms is immaterial. For example

$$a \cdot b \cdot c = b \cdot a \cdot c = c \cdot b \cdot a = \cdots.$$

This is shown by repeated application of the commutative property (together with F 1).

F 3. For any $a, b \in \mathbf{R}$ the equation $x + a = b$ has one and only one solution. For if $x \in \mathbf{R}$ is such that $x + a = b$, then $x = x + 0 = x + (a + (-a)) = (x + a) + (-a) = b + (-a)$, so $x = b + (-a)$ is the only possible solution; that this is indeed a solution is immediate. One consequence is that the element 0 of Property IV is unique; another is that for any $a \in \mathbf{R}$, the element $-a$ of Property V is unique.

For convenience, instead of $b + (-a)$ one usually writes $b - a$. (This is a definition of the symbol "$-$" between two elements of $\mathbf{R}$.) Thus $-a = 0 - a$.

We take the opportunity to reiterate here the important role of convention. $a + b + c$ has been defined (and by F 1 there is only one reasonable way to define it), but we have not yet defined $a - b - c$. Of course by the latter expression we understand $(a - b) - c$, but it is important to realize that this is merely convention, and reading aloud the words "a minus b minus c" with a sufficient pause after the first "minus" points out that our convention could equally well have defined $a - b - c$ to be $a - (b - c)$. In this connection note the absence of *any* standard convention for $a \div b \div c$. In a similar connection, note that a^{b^c} could be taken to mean $(a^b)^c$ if it were not conventionally taken to mean $a^{(b^c)}$. As stated above we use all the ordinary notational conventions when no confusion can result. For example, without further ado we shall interpret an expression like $\log a^b$ to mean $\log(a^b)$ and not $(\log a)^b$, ab^{-1} does not mean $(ab)^{-1}$, etc.

F 4. For any $a, b \in \mathbf{R}$, with $a \neq 0$, the equation $xa = b$ has one and only one solution. In fact from $xa = b$ follows $x = xaa^{-1} = ba^{-1}$, and from $x = ba^{-1}$ follows $xa = b$. Thus the element 1 of Property IV is unique and, given any $a \in \mathbf{R}$, $a \neq 0$, the element a^{-1} of Property V is unique.

For $a, b \in \mathbf{R}$, $a \neq 0$, we define b/a, in accord with convention, to be $b \cdot a^{-1}$. In particular, $a^{-1} = 1/a$.

F 5. For any $a \in \mathbf{R}$ we have $a \cdot 0 = 0$. This is true since $a \cdot 0 + a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 = a \cdot 0 + 0$, so that $a \cdot 0$ and 0 are both solutions of the equation $x + a \cdot 0 = a \cdot 0$, hence equal, by F 3. From this it follows immediately that if a product of several elements of **R** is 0 then one of the factors must be 0: for if $ab = 0$ and $a \neq 0$ we can multiply both sides by a^{-1} to get $b = 0$. Hence the illegitimacy of division by zero.

F 6. $-(-a) = a$ for any $a \in \mathbf{R}$. For both $-(-a)$ and a are solutions of the equation $x + (-a) = 0$, hence equal, by F 3.

F 7. $(a^{-1})^{-1} = a$ for any nonzero $a \in \mathbf{R}$. In fact since $a \cdot a^{-1} = 1$, by F 5 we know that $a^{-1} \neq 0$, so $(a^{-1})^{-1}$ exists, and F 4 implies that $(a^{-1})^{-1}$ and a are equal, since both are solutions of the equation $x \cdot a^{-1} = 1$.

F 8. $-(a + b) = (-a) + (-b)$ for all $a, b \in \mathbf{R}$. For both are solutions of the equation $x + (a + b) = 0$.

F 9. $(ab)^{-1} = a^{-1}b^{-1}$ if a, b are nonzero elements of **R**. For $ab \neq 0$ by F 5, so $(ab)^{-1}$ exists, and both $(ab)^{-1}$ and $a^{-1}b^{-1}$ are solutions of the equation $x(ab) = 1$.

The usual rules for operating with fractions follow easily from F 9:

$$\frac{ac}{bc} = (ac)(bc)^{-1} = acb^{-1}c^{-1} = ab^{-1} = \frac{a}{b},$$

$$\frac{a}{b} \cdot \frac{c}{d} = (ab^{-1})(cd^{-1}) = ac(bd)^{-1} = \frac{ac}{bd},$$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = (ad)(bd)^{-1} + (bc)(bd)^{-1}$$

$$= (ad + bc)(bd)^{-1} = \frac{ad + bc}{bd}.$$

F 10. $-a = (-1) \cdot a$ for all $a \in \mathbf{R}$. For $(-1) \cdot a + a = a \cdot ((-1) + 1) = a \cdot 0 = 0$, so that $(-1) \cdot a$ and $-a$ are both solutions of the equation $x + a = 0$, hence are equal. Two immediate consequences are $a \cdot (-b) = a \cdot (-1) \cdot b = (-1) \cdot a \cdot b = (-a) \cdot b = -ab$ and $(-a) \cdot (-b) = -(a \cdot (-b)) = -(-ab) = ab$.

Notice that all five field properties of the real numbers, and therefore all consequences of them, are satisfied by the rational numbers, or by the complex numbers. That is, the rational numbers and the complex numbers are also fields. In fact there exist fields with only a finite number of elements, the simplest one being a field with just the two elements 0 and 1. To describe the real numbers completely, more properties are needed.

§ 2. ORDER.

The order property of the real number system is the following:

PROPERTY VI. *There is a subset $\mathbf{R}_+$ of $\mathbf{R}$ such that*
(1) if $a, b \in \mathbf{R}_+$, then $a + b, a \cdot b \in \mathbf{R}_+$
(2) for any $a \in \mathbf{R}$, one and only one of the following statements is true

$$a \in \mathbf{R}_+$$
$$a = 0$$
$$-a \in \mathbf{R}_+.$$

The elements $a \in \mathbf{R}$ such that $a \in \mathbf{R}_+$ will of course be called *positive*, those such that $-a \in \mathbf{R}_+$ *negative*. From the above property of $\mathbf{R}_+$ we shall deduce all the usual rules for working with inequalities.

To be able to express the consequences of Property VI most conveniently we introduce the relations "$>$" and "$<$". For $a, b \in \mathbf{R}$, either of the expressions

$$a > b \quad \text{or} \quad b < a$$

(read respectively as "a is greater than b" and "b is less than a") will mean that $a - b \in \mathbf{R}_+$. Either of the expressions

$$a \geq b \quad \text{or} \quad b \leq a$$

will mean that $a > b$ or $a = b$.

Clearly $a \in \mathbf{R}_+$ if and only if $a > 0$. An element $a \in \mathbf{R}$ is negative if and only if $a < 0$.

The following are the consequences of the order property.

O 1. **(Trichotomy).** If $a, b \in \mathbf{R}$ then one and only one of the following statements is true:

$$a > b$$
$$a = b$$
$$a < b.$$

For if we apply part (2) of the order property to the number $a - b$ then exactly one of three possibilities holds, $a - b \in \mathbf{R}_+$, $a - b = 0$, or $b - a \in \mathbf{R}_+$, which are the three cases of the assertion O 1.

O 2. **(Transitivity).** If $a > b$ and $b > c$ then $a > c$. For we are given $a - b \in \mathbf{R}_+$ and $b - c \in \mathbf{R}_+$; it therefore follows that $a - c = (a - b) + (b - c) \in \mathbf{R}_+$, so $a > c$.

O 3. If $a > b$ and $c \geq d$ then $a + c > b + d$. In fact, the hypotheses mean $a - b \in \mathbf{R}_+$, $c - d \in \mathbf{R}_+ \cup \{0\}$, and as a consequence $(a + c) - (b + d) = (a - b) + (c - d) \in \mathbf{R}_+$, proving the assertion.

O 4. If $a > b > 0$ (meaning that $a > b$ and $b > 0$) and $c \geq d > 0$, then $ac > bd$. For $a - b \in \mathbf{R}_+$ and $c \in \mathbf{R}_+$, so $ac - bc = (a - b)c \in \mathbf{R}_+$, and similarly $c - d \in \mathbf{R}_+ \cup \{0\}$ and $b \in \mathbf{R}_+$ together imply that $bc - bd \in \mathbf{R}_+ \cup \{0\}$; it necessarily follows that $ac - bd = (ac - bc) + (bc - bd) \in \mathbf{R}_+$, that is $ac > bd$.

Note that the assumptions that b and d are positive are essential; the assertion O 4 does not hold, for example, with $a = 1$, $b = -1$, $c = 2$, $d = -3$.

O 5. The following rules of sign for adding and multiplying real numbers hold:

(positive number) + (positive number) = (positive number)
(negative number) + (negative number) = (negative number)
(positive number) · (positive number) = (positive number)
(positive number) · (negative number) = (negative number)
(negative number) · (negative number) = (positive number).

These are immediate from F 10 and Property VI.

O 6. For any $a \in \mathbf{R}$ we have $a^2 \geq 0$, with the equality holding only if $a = 0$; more generally the sum of the squares of several elements of $\mathbf{R}$ is always greater than or equal to zero, with equality only if all the elements in question are zero. For by O 5, the statement $a \neq 0$ implies $a^2 > 0$, and a sum of positive elements is positive. Note the special consequence $1 = 1^2 > 0$.

O 7. If $a > 0$, then $1/a > 0$. In fact $a \cdot (1/a) = 1 > 0$, which would contradict the rules of sign if we had $1/a \leq 0$.

O 8. If $a > b > 0$, then $1/a < 1/b$. For $ab > 0$, hence $(ab)^{-1} > 0$, so $(ab)^{-1}a > (ab)^{-1}b$, which simplifies to $1/b > 1/a$.

O 9. We now show how the computational rules of elementary arithmetic work out as consequences of our assumptions. Let us make the definitions $2 = 1 + 1$, $3 = 2 + 1$, $4 = 3 + 1$, etc., and let us define the *natural numbers* to be the set $\{1, 2, 3, \ldots\}$. Since $1 > 0$ it follows that $0 < 1 < 2 < 3 < \cdots$. The set of natural numbers is ordered exactly as we would like it to be—in particular, the natural numbers have the following properties: for any natural numbers a, b, exactly one of the statements $a < b$, $a = b$, $b < a$ holds; if a, b, c are natural numbers and $a < b$ and $b < c$ then also $a < c$; any natural number has an immediate successor (a least natural number that is greater than it); different natural numbers have different immediate successors; and there is a natural number 1 with the property that any set of natural numbers that includes 1 and with each element also its immediate successor consists of all natural numbers. For any natural number n, n is the sum of a set of 1's that is in one-one correspondence with the elements of the set $\{1, 2, 3, \ldots, n\}$. This implies that in whatever order we count off the elements of a set of n objects (that is, a set in one-one correspondence with the set $\{1, 2, 3, \ldots, n\}$) we arrive at the final count n, and if a proper subset of a set of n objects has m objects, then $m < n$. The usual rules for adding natural numbers come from such computations as

$$2 + 3 = (1 + 1) + (1 + 1 + 1) = 1 + 1 + 1 + 1 + 1 = 5,$$

while the rules for multiplication follow from the fact that sums of equal terms may be written as products; for example, for any $a \in \mathbf{R}$ we have $a + a + a = (1 + 1 + 1) \cdot a = 3a$. Thus $3 \cdot 4 = 4 + 4 + 4 = 12$, so we can verify the entire multiplication table, as high as we care to go. The *integers*, that is the subset $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$ of $\mathbf{R}$, are also ordered in the correct way $\cdots < -2 < -1 < 0 < 1 < 2 < \cdots$. It is easy to check that the integers add according to the ordinary rules; that they multiply in the usual way is implied by F 10 and the corresponding fact for the natural numbers. The *rational numbers*, that is the elements of $\mathbf{R}$ which can be written a/b, with a, b integers and $b \neq 0$, are also ordered in the usual way; indeed the order relation of two rational numbers can be determined by writing the two numbers with a positive common denominator and comparing the numerators. Addition and multiplication of rational numbers are also determined by the same operations for the integers. Thus the rational numbers, a certain subset of $\mathbf{R}$, have all the arithmetic and order properties with which we are familiar.

Here is as good a place as any to introduce into our logical discussion of the real number system the notion of exponentiation with integral exponents. If $a \in \mathbf{R}$ and n is some positive integer we define a^n to be

$a \cdot a \cdot a \cdots a$ (n times), and if $a \neq 0$ we define $a^0 = 1$, $a^{-n} = 1/a^n$. From these definitions we immediately derive the usual rules of exponentiation, in particular

$$a^m \cdot a^n = a^{m+n}$$
$$(a^m)^n = a^{mn}$$
$$(ab)^n = a^n b^n$$

The definition of the absolute value of a real number is most conveniently introduced at this point: if $a \in \mathbf{R}$, the *absolute value of a*, denoted $|a|$, is given by

$$\begin{aligned} |a| &= a \quad \text{if} \quad a > 0, \\ |a| &= 0 \quad \text{if} \quad a = 0, \\ |a| &= -a \quad \text{if} \quad a < 0. \end{aligned}$$

The absolute value has the following properties:

(1) $|a| \geq 0$ for all $a \in \mathbf{R}$, and $|a| = 0$ if and only if $a = 0$
(2) $|ab| = |a| \cdot |b|$ for all $a, b \in \mathbf{R}$
(3) $|a|^2 = a^2$ for all $a \in \mathbf{R}$
(4) $|a + b| \leq |a| + |b|$ for all $a, b \in \mathbf{R}$
(5) $|a - b| \geq \big||a| - |b|\big|$ for all $a, b \in \mathbf{R}$.

The first three properties above are trivial consequences of the definition of $|a|$. To prove (4) note first that

$$\pm a \leq |a|$$

(meaning that $a \leq |a|$ and $-a \leq |a|$) and

$$\pm b \leq |b|,$$

so adding gives

$$\pm(a + b) \leq |a| + |b|,$$

or

$$|a + b| \leq |a| + |b|.$$

To prove (5), note that $|a| = |(a - b) + b| \leq |a - b| + |b|$, so that

$$|a - b| \geq |a| - |b|.$$

Interchanging a and b,

$$|a - b| \geq |b| - |a|,$$

and the last two inequalities combine into (5).

It is useful to note that repeated application of (4) gives

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

We also note the trivial but very useful fact that if $x, a, \epsilon \in \mathbf{R}$, then

$$|x - a| < \epsilon$$

if and only if

$$a - \epsilon < x < a + \epsilon.$$

For $|x - a| < \epsilon$ is precisely equivalent to $x - a < \epsilon$ and $-(x - a) < \epsilon$, or $-\epsilon < x - a < \epsilon$, which in turn is equivalent to $a - \epsilon < x < a + \epsilon$.

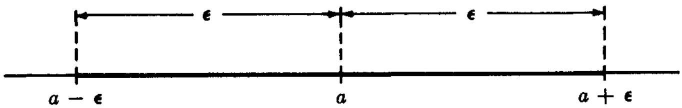

FIGURE 5. The points x such that $|x - a| < \epsilon$.

At the end of the previous section a number of other systems were given which satisfy the first five properties of the real number system. The order property excludes two of the systems given there: the field consisting of just the two elements 0 and 1 (since then $1 + 1 = 0$, contradicting $1 + 1 > 0$), and the complex numbers (since any number must have a nonnegative square). But the rational numbers satisfy all the properties given so far. Since it is known that there exist real numbers which are not rational (this will be proved shortly), still more properties are needed to describe the real numbers completely.

§ 3. THE LEAST UPPER BOUND PROPERTY.

To introduce the last fundamental property of the real number system we need the following concepts. If $S \subset \mathbf{R}$, then *an upper bound for the set* S is a number $a \in \mathbf{R}$ such that $s \leq a$ for each $s \in S$. If the set S has an upper bound, we say that S is *bounded from above.* We call a real number y a *least upper bound of the set* S if

(1) y is an upper bound for S

and

(2) if a is any upper bound for S, then $y \leq a$.

From this definition it follows that two least upper bounds of a set $S \subset \mathbf{R}$ must be less than or equal to each other, hence equal. Thus a set $S \subset \mathbf{R}$ can have at most one least upper bound and we may speak of *the* least upper bound of S (if one exists). Note also the following important fact: if y is the least upper bound of S and $x \in \mathbf{R}$, $x < y$, then there exists an element $s \in S$ such that $x < s$.

A nonempty finite subset $S \subset \mathbf{R}$ always has a least upper bound; in this case the least upper bound is simply the greatest element of S. More generally any subset $S \subset \mathbf{R}$ that has a greatest element (usually denoted max S) has max S as a least upper bound. But an infinite subset of $\mathbf{R}$ need not have a least upper bound, for example, $\mathbf{R}$ itself has no upper bound at all. Furthermore, if a subset S of $\mathbf{R}$ has a least upper bound it does not necessarily follow that this least upper bound is in S; for example, if S is the set of all negative numbers then S has no greatest element, but any $a \geq 0$ is an upper bound of S and zero (a number *not* in S) is the least upper bound of S.

The last axiom for the real number system is the following, which gives a further condition on the ordering of Property VI.

PROPERTY VII. (LEAST UPPER BOUND PROPERTY). *A nonempty set of real numbers that is bounded from above has a least upper bound.*

If we look at the real numbers geometrically, imagining them plotted on a straight line in the usual manner of analytic geometry, Property VII becomes quite plausible. For if $S \subset \mathbf{R}$ is nonempty and bounded from above then either S has a greatest element or, if we try to pick a point in S as far to the right as possible, we can find a point in S such that no point in S is more than a distance of one unit to the right of the chosen point. Then we can pick a point in S farther to the right than the first chosen point and such that no point in S is more than one-half unit to the right of this second chosen point, then a point of S still farther to the right such that no point of S is more than one-third unit to the right of the last chosen point, etc. It is intuitively clear that the sequence of chosen points in S must "gang up" toward some point of $\mathbf{R}$, and this last point will be the least upper bound of S. (See Figure 6.)

Another way to justify Property VII in our minds is to look upon the real numbers as represented by infinite decimals, i.e., symbols of the form

$$(\text{integer}) + .a_1a_2a_3\ldots,$$

where each of the symbols $a_1, a_2, a_3, \ldots$ is one of the integers $0, 1, 2, \ldots, 9$, with the symbols $<, >, +, \cdot$ being interpreted for infinite decimals in the standard way. (Note that any terminating decimal can be considered an

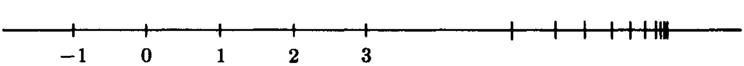

FIGURE 6. A sequence of points in $\mathbf{R}$ ganging up toward a least upper bound.

infinite decimal by adding an infinite string of zeros.) If S is a nonempty set of infinite decimals that is bounded from above, then we can find an element of S whose integral part is maximal, then an element of S having the same integral part and with a_1 maximal, then an element of S having the same integral part and same a_1 with a_2 maximal, and we can continue this process indefinitely, ending up with an infinite decimal (which may or may not be in S) which is clearly a least upper bound of S.*

The least upper bound of a subset S of $\mathbf{R}$ will be denoted l.u.b. S; another common notation is sup S (sup standing for the Latin *supremum*). Property VII says that l.u.b. S exists whenever $S \subset \mathbf{R}$ is nonempty and bounded from above. Conversely, if $S \subset \mathbf{R}$ and l.u.b. S exists, then S must be nonempty (for *any* real number is an upper bound for the empty set and there is no least real number) and bounded from above.

Analogous to the above there are the notions of lower bound and greatest lower bound: $a \in \mathbf{R}$ is a *lower bound* for the subset $S \subset \mathbf{R}$ if $a \leq s$ for each $s \in S$, and a is a *greatest lower bound of S* if a is a lower bound of S and there exists no larger one. S is called *bounded from below* if it has a lower bound. It follows from Property VII that every set S of real numbers that is nonempty and bounded from below has a greatest lower bound: as a matter of fact, a set $S \subset \mathbf{R}$ is bounded from below if and only if the set $S' = \{x : -x \in S\}$ is bounded from above, and if S is nonempty and bounded from below then $-$l.u.b. S' is the greatest lower bound of S. The greatest lower bound of a subset S of $\mathbf{R}$ is denoted g.l.b. S; another notation is inf S (inf abbreviating the Latin *infimum*). If S has a smallest element (for example, if S is finite and nonempty) then g.l.b. S is simply this smallest element, often denoted min S.

We proceed to draw some consequences of Property VII. Among other things we shall show that the real numbers are not very far from the rational numbers, in the sense that any real number may be "approximated as closely as we wish" by rational numbers. The way to view the situation is that the rational numbers are in many ways very nice, but there are certain "gaps" among them that may prevent us from doing all the things we would like to do with numbers, such as solving equations (e.g., extracting roots), or measuring geometric objects, and the introduction of the real numbers that are not rational amounts to closing the gaps.

Here are the consequences of the least upper bound property:

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* Let us remark here that once the set of integers is known, together with their addition and multiplication, it is possible to construct the real number system by defining real numbers by means of infinite decimals. This is in fact the way real numbers are usually introduced in elementary arithmetic, and we know how easy it is to compute with decimals. But there are a few inconveniences in this method stemming from the fact that some numbers have more than one decimal representation (e.g., .999 ... = 1.000 ...). There is also the esthetic inconvenience of giving a preferred status to the number 10—almost a biological accident. In any case we shall discuss later in this section how the seven properties of real numbers imply that they can indeed be represented by infinite decimals, thus completing the circle with elementary arithmetic.

LUB 1. For any real number x, there is an integer n such that $n > x$. (In other words, there exist arbitrarily large integers.) To prove this, assume we have a real number x for which the assertion is wrong. Then $n \leq x$ for each integer n, so that the set of integers is bounded from above. Since the set of integers is nonempty it has a least upper bound, say a. But for any integer n, $n + 1$ is also an integer, so $n + 1 \leq a$ and thus $n \leq a - 1$, showing that $a - 1$ is also an upper bound for the set of integers. Since $a - 1 < a$, a is not a least upper bound. This is a contradiction.

LUB 2. For any positive real number ϵ there exists an integer n such that $1/n < \epsilon$. (In other words, there are arbitrarily small positive rational numbers.) For the proof it suffices to choose an integer $n > 1/\epsilon$, which is possible by LUB 1, then use O 8, which is permissible since by O 7 we have $1/\epsilon > 0$.

LUB 3. For any $x \in \mathbf{R}$ there is an integer n such that $n \leq x < n + 1$. To prove this, choose an integer $N > |x|$, so that $-N < x < N$. The integers from $-N$ to N form the finite set $\{-N, -N + 1, \ldots, 0, 1, \ldots, N\}$ and all we need do is take n to be the greatest of these that is less than or equal to x.

LUB 4. For any $x \in \mathbf{R}$ and positive integer N, there is an integer n such that

$$\frac{n}{N} \leq x < \frac{n+1}{N}.$$

To show this we merely have to apply LUB 3 to the number Nx, getting an integer n such that $n \leq Nx < n + 1$.

LUB 5. If $x, \epsilon \in \mathbf{R}$, $\epsilon > 0$, then there exists a rational number r such that $|x - r| < \epsilon$. (In other words, a real number may be approximated as closely as we wish by a rational number.) To prove this, use LUB 2 to find a positive integer N such that $1/N < \epsilon$, then use LUB 4 to find an integer n such that $n/N \leq x < (n + 1)/N$. Then $0 \leq x - n/N < 1/N < \epsilon$, so $|x - n/N| < \epsilon$.

We now discuss the decimal representation of real numbers. First consider finite decimals. If a_0 is any integer, n any positive integer, and $a_1, a_2, \ldots, a_n$ any integers chosen from among $0, 1, 2, \ldots, 9$, the symbol

$$a_0.a_1\,a_2\,\ldots\,a_n$$

will mean, as usual, the rational number

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n}.$$

If m is a positive integer less than n, then

$$a_0.a_1\ldots a_m \leq a_0.a_1\ldots a_n = a_0.a_1\ldots a_m + a_{m+1}\cdot 10^{-(m+1)} + \cdots + a_n \cdot 10^{-n}$$
$$\leq a_0.a_1\ldots a_m + 9 \cdot 10^{-(m+1)} + \cdots + 9 \cdot 10^{-n}.$$

If we add 10^{-n} to this last number a lot of cancellation occurs, resulting in

$$a_0.a_1\ldots a_m \leq a_0.a_1\ldots a_n < a_0.a_1\ldots a_m + 10^{-m}.$$

This last inequality is at the base of most rounding-off procedures in approximate calculations and in addition shows that two numbers in the above decimal form are equal only if (except for the possible addition of a number of zeros to the right, which doesn't change the value of the symbol) they have the same digits in corresponding places. It also enables us to tell at a glance which of two numbers in the given form is larger. The ordinary rules for adding and multiplying numbers in this form are clearly legitimate.

By an *infinite decimal* we mean a formal expression

$$a_0.a_1a_2a_3\ldots$$

(this is just another way of writing a sequence) where a_0 is an integer and each of $a_1, a_2, a_3, \ldots$ is one of the integers $0, 1, \ldots, 9$. The set $\{a_0.a_1\ldots a_n : n = \text{positive integer}\}$ is nonempty and bounded from above (for any integer $m > 0$, $a_0.a_1\ldots a_m + 10^{-m}$ is an upper bound) hence has a least upper bound. The symbol $a_0.a_1a_2a_3\ldots$ is called a *decimal expansion* for this least upper bound and we say that the least upper bound is *represented by* the infinite decimal. Thus every infinite decimal is a decimal expansion for a definite real number and we may use the infinite decimal itself as a symbol for the number. Thus

$$a_0.a_1a_2a_3\ldots = \text{l.u.b. } \{a_0.a_1\ldots a_n : n = \text{positive integer}\},$$

and for any positive integer n we have the inequality

$$a_0.a_1\ldots a_n \leq a_0.a_1a_2a_3\ldots \leq a_0.a_1\ldots a_n + 10^{-n}.$$

This enables us to tell immediately which of two infinite decimals represents the larger real number. Note that two different infinite decimals may be decimal expansions for the same real number, for example $5.1399999\ldots = 5.1400000\ldots$, but the last inequality shows that different infinite decimals are decimal expansions for the same real number only in this case, that is when we can get one infinite decimal from the other by replacing one of the digits $0, 1, \ldots, 8$ followed by an infinite sequence of nines by the next higher digit followed by a sequence of zeros.

Any real number is represented by at least one infinite decimal. To see this, apply LUB 4 to the case $N = 10^m$, where m is any positive integer: we get a finite decimal $a_0.a_1\ldots a_m$ such that

$$a_0.a_1\ldots a_m \leq x < a_0.a_1\ldots a_m + 10^{-m}.$$

If we try doing this for $m + 1$ in place of m, then a_0 and the digits $a_1, \ldots, a_m$ will not change, and we simply get another digit a_{m+1}. Letting m get larger and larger, we get more and more digits of an infinite decimal, and this is our desired decimal expansion for x. Note that the addition or multiplication of two infinite decimals goes according to the usual rules: we round off each decimal and add or multiply the corresponding finite decimals to get a decimal approximation of the desired sum or product. We obtain as many digits as we wish of the decimal expansion of the sum or product by rounding off the given infinite decimals to a sufficiently large number of places.

Using decimal expansions of real numbers, it is very easy to exhibit real numbers which are not rational. One such number is

$$.101001000100001000001\ldots .$$

Multiply this by any positive integer and one gets a number which is not an integer, so this number cannot be rational.

§ 4. THE EXISTENCE OF SQUARE ROOTS.

It is convenient to prove here a special result, even though this can be derived as a consequence of a much more general theorem to be proved later.

A square root of a given number is a number whose square is the given number. Since the square of any nonzero number is positive, only nonnegative numbers can have square roots. The number zero has one square root, which is zero itself.

Proposition. *Every positive number has a unique positive square root.*

If $0 < x_1 < x_2$ then $x_1^2 < x_2^2$. That is, bigger positive numbers have bigger squares. Thus any given real number can have at most one positive square root. It remains to show that if $a \in \mathbf{R}$, $a > 0$, then a has at least one positive square root. For this purpose consider the set

$$S = \{x \in \mathbf{R} : x \geq 0,\ x^2 \leq a\}.$$

This set is nonempty, since $0 \in S$, and bounded from above, since if $x > \max\{a, 1\}$ we have $x^2 = x \cdot x > x \cdot 1 = x > a$. Hence $y = \text{l.u.b. } S$ exists. We proceed to show that $y^2 = a$. First, $y > 0$, for $\min\{1, a\} \in S$, since $(\min\{1, a\})^2 \leq \min\{1, a\} \cdot 1 = \min\{1, a\} \leq a$. Next, for any ϵ such that $0 < \epsilon < y$ we have $0 < y - \epsilon < y < y + \epsilon$, so

$$(y - \epsilon)^2 < y^2 < (y + \epsilon)^2,$$

since bigger positive numbers have bigger squares. By the definition of y there are numbers greater than $y - \epsilon$ in S, but $y + \epsilon \notin S$. Again using the fact that bigger positive numbers have bigger squares, we get

$$(y - \epsilon)^2 < a < (y + \epsilon)^2.$$

Hence

$$(y-\epsilon)^2-(y+\epsilon)^2<y^2-a<(y+\epsilon)^2-(y-\epsilon)^2,$$

so

$$|y^2-a|<(y+\epsilon)^2-(y-\epsilon)^2=4y\epsilon.$$

The inequality $|y^2-a|<4y\epsilon$ holds for any ϵ such that $0<\epsilon<y$, and by choosing ϵ small enough we can make $4y\epsilon$ less than any preassigned positive number. Thus $|y^2-a|$ is less than any positive number. Since $|y^2-a|\geq 0$, we must have $|y^2-a|=0$, proving $y^2=a$.

If $a>0$, the unique positive square root of a is denoted $\sqrt{a}$; thus a has exactly two square roots, namely $\sqrt{a}$ and $-\sqrt{a}$. We also write $\sqrt{0}=0$.

We now know that the positive real numbers are precisely the squares of the nonzero real numbers. This shows that the set of positive numbers $\mathbf{R}_+$ whose existence is affirmed by Property VI is completely determined by the multiplication function of $\mathbf{R}$. A priori, it might seem that there could be several possible subsets $\mathbf{R}_+$ of $\mathbf{R}$ for which Properties VI and VII hold and that in any discussion of the ordering of $\mathbf{R}$ the subset $\mathbf{R}_+$ would have to be specified, but we now know this to be unnecessary. The set $\mathbf{R}$, together with the functions $+$ and $\cdot$, determine the ordering of $\mathbf{R}$. It therefore follows that the decimal expansions of elements of $\mathbf{R}$ are completely determined by the triple $\{\mathbf{R}, +, \cdot\}$. Since the addition and multiplication of decimals follow the usual rules of arithmetic, *the real number system is completely determined by Properties I–VII*, in the sense that if we have another triple $\{\mathbf{R}', +', \cdot'\}$ satisfying these properties then there will exist a unique one-one correspondence between $\mathbf{R}$ and $\mathbf{R}'$ preserving sums and products. Thus we may speak of *the* real number system. In fact one often speaks of "the real numbers $\mathbf{R}$", meaning the real number *system*; this is strictly speaking erroneous, since $\mathbf{R}$ is merely a set and we also have to know what the operations $+$ and $\cdot$ on this set are, but when there is no danger of confusion this is a convenient abbreviation.

PROBLEMS

1. Show that there exists one and (essentially) only one field with three elements.
2. Prove in detail that for any $a, b, c, d \in \mathbf{R}$
   (a) $-(a-b)=b-a$
   (b) $(a-b)(c-d)=(ac+bd)-(ad+bc)$.
3. Prove that if $a, b \in \mathbf{R}$ and $a<b<0$, then $1/a>1/b$.
4. (a) Is 223/71 greater than 22/7?
   (b) Is 265/153 greater than 1351/780?

5. For which $x \in \mathbf{R}$ are the following inequalities true?
   (a) $3(x+2) < x+5$
   (b) $x^2 - 5x - 6 \geq 0$
   (c) $\frac{2}{x} > x - 1$
   (d) $\frac{7}{x-3} > x + 3 > 0.$

6. Show that if $a, b, x, y \in \mathbf{R}$ and $a < x < b$, $a < y < b$, then $|y - x| < b - a$.

7. Show that for any $a, b \in \mathbf{R}$,

$$\max\{a, b\} = \frac{a + b + |a - b|}{2}$$

$$\min\{a, b\} = -\max\{-a, -b\} = \frac{a + b - |a - b|}{2}.$$

8. The *complex number system* is defined to be the set $\mathbf{C} = \mathbf{R} \times \mathbf{R}$ (called the *complex numbers*) together with the two functions from $\mathbf{C} \times \mathbf{C}$ into $\mathbf{C}$, denoted by $+$ and $\cdot$, that are given by $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$ for all $a, b, c, d \in \mathbf{R}$.
   (a) Show that $\mathbf{C}$, together with the functions $+$ and $\cdot$, is a field.
   (b) Show that the map from $\mathbf{R}$ into $\mathbf{C}$ which sends each $a \in \mathbf{R}$ into $(a, 0)$ is one-one and "preserves addition and multiplication" (being careful to define the meaning of the words in quotes).
   (c) Identifying $\mathbf{R}$ with a subset of $\mathbf{C}$ by means of part (b) (so that we can consider $\mathbf{R} \subset \mathbf{C}$) and setting $i = (0, 1)$, show that $i^2 = -1$ and that each element of $\mathbf{C}$ can be written in a unique way as $a + bi$, with $a, b \in \mathbf{R}$.

9. Is the subset $\varnothing$ of $\mathbf{R}$ bounded from above or below? Does it have an l.u.b. or a g.l.b.?

10. Find the g.l.b. and l.u.b. of the following sets, giving reasons if you can.
    (a) $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$
    (b) $\left\{\frac{1}{3}, \frac{4}{9}, \frac{13}{27}, \frac{40}{81}, \ldots\right\}$
    (c) $\left\{\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots\right\}.$

11. Prove that if $a \in \mathbf{R}$, $a > 1$, then the set $\{a, a^2, a^3, \ldots\}$ is not bounded from above. $\left(\textit{Hint:}\text{ First find a positive integer } n \text{ such that } a > 1 + \frac{1}{n} \text{ and prove that } a^n > \left(1 + \frac{1}{n}\right)^n \geq 2.\right)$

12. Let X and Y be nonempty subsets of $\mathbf{R}$ whose union is $\mathbf{R}$ and such that each element of X is less than each element of Y. Prove that there exists $a \in \mathbf{R}$ such that X is one of the two sets

$$\{x \in \mathbf{R} : x \leq a\} \quad \text{or} \quad \{x \in \mathbf{R} : x < a\}.$$

13. If S_1, S_2 are nonempty subsets of $\mathbf{R}$ that are bounded from above, prove that

$$\text{l.u.b. } \{x + y : x \in S_1, y \in S_2\} = \text{l.u.b. } S_1 + \text{l.u.b. } S_2.$$

14. Let $a, b \in \mathbf{R}$, with $a < b$. Show that there exists a number $x \in \mathbf{R}$ such that $a < x < b$, with x rational or not rational, as we wish.

15. "A real number is rational if and only if it has a periodic decimal expansion." Define the present usage of the word *periodic* and prove the statement.

16. Decimal (10-nary) expansions of real numbers were defined by special reference to the number 10. Show that real numbers have b-nary expansions with analogous properties, where b is any integer greater than 1.