Def: V, W two vector spaces. $T: V \to W$ is a linear transformation if

$T(f+g) = T(f) + T(g)$ for all $f, g \in V$

and $T(\lambda f) = \lambda T(f)$ for all $f \in V$, $\lambda \in \mathbb{R}$.

$Im(T) = \{ g \in W : g = T(f) \text{ for some } f \in V \}$.

$Ker(T) = \{ f \in V : T(f) = 0 \}$

$rank(T) = dim(Im(T))$

$nullity(T) = dim(Ker(T))$

If $dim V$ is finite. then

$$dim(Ker(T)) + dim(Im(T)) = dim V$$

Examples 1) $C^\infty = \{ f: \mathbb{R} \to \mathbb{R} :$ all the deribatives of f exist$\}$. $D: C^\infty \longrightarrow C^\infty$

$D(f) = f'$

$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$

$D(\lambda f) = (\lambda f)' = \lambda f' = \lambda D(f)$

then D is a linear transformation.

$Ker(D) = \{ f(x) = c \text{ for all } x \in \mathbb{R} \text{ for some } c \in \mathbb{R} \}$

Basis of $Ker(D)$ $\{1\}$ $dim(Ker(D)) = 1$

$\mathrm{Im}(D) = C^\infty$. If $f(x) \in C^\infty$, then $f(x) = D\left(\int_0^x f(t)\,dt\right)$

$\dim(C^\infty) = \infty$

If $\dim(C^\infty) < \infty$, then

$$\dim(C^\infty) = \underbrace{\dim(\mathrm{Ker}(D))}_{=1} + \underbrace{\dim(\mathrm{Im}(D))}_{\dim(C^\infty)}$$

2) $C[0,1] = \{ f : [0,1] \to \mathbb{R} : f \text{ is continuous} \}$

$I : C[0,1] \longrightarrow \mathbb{R}$

$I(f) = \int_0^1 f(x)\,dx$

$I(f+g) = I(f) + I(g)$ & $I(\lambda f) = \lambda I(g)$

Def: V, W vector spaces. $T: V \to W$ linear transformation from V to W. We say that T is an isomorphism if $\mathrm{Ker}(T) = \{0\}$ and $\mathrm{Im}(T) = W$. this is the same, as saying that T is a bijection, i.e one to one and onto. In this case, we say that V & W are isomorphic.

Obs: Let V be a vector space. Let $\{f_1, \dots, f_n\}$ be a basis of V. $\mathcal{B} = \{f_1, \dots, f_n\}$. $L_{\mathcal{B}} : V \to \mathbb{R}^n$.

$L_{\mathcal{B}}(f) = [f]_{\mathcal{B}}$. $L_{\mathcal{B}}$ is an isomorphism.

~~proof~~: $L_{\mathcal{B}}(f+g) = [f+g]_{\mathcal{B}} = [f]_{\mathcal{B}} + [g]_{\mathcal{B}} = L_{\mathcal{B}}(f) + L_{\mathcal{B}}(g)$

Recall $L_{\mathcal{B}}(f) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ if $f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$

$L_{\mathcal{B}}(\lambda f) = \lambda L_{\mathcal{B}}(f)$

$L_{\mathcal{B}}(f) = 0 \Rightarrow [f]_{\mathcal{B}} = 0 \Rightarrow f = 0 \cdot f_1 + 0 \cdot f_2 + \cdots + 0 \cdot f_n$

then $f = 0$. then $\ker(L_{\mathcal{B}}) = \{0\}$

Let $c = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$ $L_{\mathcal{B}}(c_1 f_1 + c_2 f_2 + \cdots + c_n f_n) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

then $\mathrm{Im}(L_{\mathcal{B}}) = \mathbb{R}^n$

<u>Ex</u>: $T: \mathbb{R}^{2\times 2} \longrightarrow \mathbb{R}^{2\times 2}$ Let $S \in \mathbb{R}^{2\times 2}$ invertible.

$T(A) = S^{-1}AS$. Show that T is a linear transformation

$T(A+B) = S^{-1}(A+B)S = S^{-1}(AS+BS) = S^{-1}AS + S^{-1}BS =$

$= T(A) + T(B)$ ✓

$T(\lambda A) = S^{-1}(\lambda A)S = \lambda S^{-1}AS = \lambda T(A)$ ✓

What is $\ker(T)$?

$T(A) = 0$ $\quad S^{-1}AS = 0$ $\quad SS^{-1}AS = S0$

$AS = 0$ $\quad ASS^{-1} = 0S^{-1}$ $\quad A = 0$ ✓

Then $\mathrm{Ker}(T) = \{0\}$.

What is $\mathrm{Im}(T)$? Claim $\mathrm{Im}(T) = \mathbb{R}^{2\times 2}$.

Let $B \in \mathbb{R}^{2\times 2}$. Is $B = T(A)$ for some A?

$B = T(A) \qquad B = S^{-1}AS \qquad SBS^{-1} = A$

$B = T(SBS^{-1})$ ✓

Basis of $\mathbb{R}^{2\times 2}$

$A \in \mathbb{R}^{2\times 2}$ then $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} +$

$c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

then $\mathbb{R}^{2\times 2} = \mathrm{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

Are $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ linearly independent? Yes because

$$\underbrace{a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ implies $\begin{matrix} a = 0 & c = 0 \\ b = 0 & d = 0 \end{matrix}$ ✓

Theorem: $T: V \to W$ linear transformation. $\dim V$ and $\dim W$ are finite. Then

1) V and W are isomorphic if and only if $\dim V = \dim W$

2) If $\operatorname{Ker}(T) = \{0\}$ and $\dim V = \dim W$, then T is an isomorphism.

3) If $\operatorname{Im}(T) = W$ and $\dim V = \dim W$, then T is an isomorphism.

~~proof~~: 1) Assume $\dim V = \dim W = n$

Let $\{f_1, \dots, f_n\}$ be a basis of V

Let $\{g_1, \dots, g_n\}$ be a basis of W

$T(f_i) = g_i \qquad 1 \leq i \leq n.$

$f \in V$, then $f = c_1 f_1 + \dots + c_n f_n$

$T(f) = c_1 T(f_1) + \dots + c_n T(f_n) = c_1 g_1 + \dots + c_n g_n$

Claim T is linear transformation.

Let $f, h \in V$. $\quad f = c_1 f_1 + \dots + c_n f_n$

$h = d_1 f_1 + \dots + d_n f_n$

$f + h = (c_1 + d_1) f + \dots + (c_n + d_n) f_n$

then $T(f) = c_1 g_1 + \dots + c_n g_n$

$T(h) = d_1 g_1 + \dots + d_n g_n$

$T(f+h) = (c_1 + d_1) g_1 + \dots + (c_n + d_n) g_n = T(f) + T(h).$

Similarly $T(\lambda f) = \lambda T(f)$

$T(f) = 0$ implies $c_1 g_1 + \dots + c_n g_n = 0$

Since $\{g_1, \dots, g_n\}$ is a basis, then $c_1 = \dots = c_n = 0$.

then $f = c_1 f_1 + \dots + c_n f = 0$ thus $\ker(T) = \{0\}$

IS $\operatorname{Im}(T) = W$? Let $g \in W$. $g = c_1 g_1 + \dots + c_n g_n$

$g = T(c_1 f_1 + \dots + c_n f_n)$. then $\operatorname{Im}(T) = W$.

Matrix of a linear transformation

Ex: $P_2 = \{$ polynomials of degree 2 or less or polynomial 0 $\}$

$P_2 = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$

Basis of $P_2 = \{1, x, x^2\}$

$T : P_2 \to P_2$

$T(f) = f' + f''$ $\qquad$ f' = derivative of f.

$f = ax^2 + bx + c$ then $f' = 2ax + b$

T is a linear transformation.

$\mathcal{B} = \{x^2, x, 1\}$ a basis of $V = P_2$

Let $S : \mathbb{R}^n \to \mathbb{R}^n$ $\quad$ $n = 3$ in this case. In general

$n = \dim V$.

$$\begin{tikzcd} P_2 \arrow[r, "T"] \arrow[d, "L_\mathcal{B}"'] & P_2 \arrow[d, "L_\mathcal{B}"] \\ {} \arrow[u, bend left, "L_\mathcal{B}^{-1}"] \arrow[r, "T"] & {} \arrow[d, "L_\mathcal{B}"] \\ {} & {} \end{tikzcd}$$

$L_\mathcal{B}$ is an isomorphism.

$L_\mathcal{B}$ is a linear transformation

$$\begin{tikzcd} {} \arrow[d, "L_{\mathcal{B}}"'] \arrow[d, bend right, shift right=2, leftarrow, "L_{\mathcal{B}}^{-1}"'] \arrow[rr, "T"] & & {} \arrow[d, bend left, shift left=2, "L_{\mathcal{B}}"] \arrow[d, "L_{\mathcal{B}}"] \\ \mathbb{R}^3 \arrow[rr, "S"'] & & \mathbb{R}^3 \end{tikzcd}$$

$L_{\mathcal{B}}$ is a linear transformation that is one-to-one and onto

$$S = L_{\mathcal{B}} T L_{\mathcal{B}}^{-1} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$Sx = Bx.$$

$$S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = L_{\mathcal{B}} T L_{\mathcal{B}}^{-1}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = L_{\mathcal{B}} T\,(ax^2+bx+c) =$$

$$= L_{\mathcal{B}}\,(2ax+b+2a) = \begin{bmatrix} 0 \\ 2a \\ 2a+b \end{bmatrix}$$

$$S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2a \\ 2a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = B\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

<u>Def</u>: $T : V \longrightarrow V$. T linear transformation. V vector space. $\dim V = n$. $\mathcal{B} = \{f_1, \dots, f_n\}$ a basis of V. $S = L_{\mathcal{B}} T L_{\mathcal{B}}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$. S is a linear transformation. $S(x) = Bx$ for all x in $\mathbb{R}^n$ and some matrix B. The matrix B is called the $\mathcal{B}$-matrix of the transformation T.

$(a)\ \forall f,\ f \in V \quad [T(f)] = B\,[f]$

Obs: Let $f \in V$ $\quad [T(f)]_{\mathcal{B}} = B\,[f]_{\mathcal{B}}$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "L_{\mathcal{B}} = [\ ]_{\mathcal{B}}"'] & V \arrow[d, "{[\ ]_{\mathcal{B}} = L_{\mathcal{B}}}"] \\ \mathbb{R}^n \arrow[r, "Bx"] & {} \end{tikzcd}$$

Obs: The j^{th} column of B is $\quad (\mathcal{B} = \{f_1, \dots, f_n\})$

$$B e_j = B\,[f_j]_{\mathcal{B}} = [T(f_j)]_{\mathcal{B}}$$

$$B = \Big[[T(f_1)]_{\mathcal{B}} \ [T(f_2)]_{\mathcal{B}} \cdots\cdots [T(f_n)]_{\mathcal{B}} \Big]$$

Ex: $T(M) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} M - M \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad T: \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Compute the $\mathcal{B}$-matrix of T.

$$\begin{tikzcd} \mathbb{R}^{2\times2} \arrow[r, "T"] \arrow[d, "L_{\mathcal{B}}"'] & \mathbb{R}^{2\times2} \arrow[d, "L_{\mathcal{B}}"] \\ \mathbb{R}^4 \arrow[r] & \mathbb{R}^4 \\ x \arrow[r, mapsto] & Bx \end{tikzcd}$$

$x \in \mathbb{R}^4$

$$L_{\mathcal{B}}^{-1}(x) = x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$$

$$x \longmapsto Bx$$

$$T(L_{\mathcal{B}}^{-1}(x)) = T\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} - \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} x_3 & x_4 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & x_1 \\ 0 & x_3 \end{bmatrix} = \begin{bmatrix} x_3 & x_4 - x_1 \\ 0 & -x_3 \end{bmatrix}$$

$$L_{\mathcal{B}}(T(L_{\mathcal{B}}^{-1}(x))) = L_{\mathcal{B}}\left(\begin{bmatrix} x_3 & x_4 - x_1 \\ 0 & -x_3 \end{bmatrix}\right) = \begin{bmatrix} x_3 \\ x_4 - x_1 \\ 0 \\ -x_3 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{\substack{\shortparallel \\ B}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Let V be a vector space. $\dim V = n$.

Let $\mathcal{B} = \{f_1, \dots, f_n\}$ be a basis of V

Let $\mathcal{U} = \{g_1, \dots, g_n\}$ be a basis of V

Let $f \in V$. Question: can we get $[f]_{\mathcal{U}}$ if we have $[f]_{\mathcal{B}}$?

$$\begin{tikzcd} V \arrow[r, "L_{\mathcal{B}}"] & \mathbb{R}^n \\ f \arrow[r] & {[f]_{\mathcal{B}}} \end{tikzcd}$$

$$L_{\mathcal{U}}(L_{\mathcal{B}}^{-1}[f]_{\mathcal{B}}) = [f]_{\mathcal{U}}$$

$$\begin{tikzcd} V \arrow[d, "L_{\mathcal{U}}"'] & f \arrow[r] \arrow[d] & {[f]_{\mathcal{B}}} \arrow[l, bend right, "L_{\mathcal{B}}^{-1}"'] \\ \mathbb{R}^n & {[f]_{\mathcal{U}}} & \end{tikzcd}$$

$L_{\mathcal{U}}$

$$L_{\mathcal{U}} \circ L_{\mathcal{B}}^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$L_{\mathcal{U}} \circ L_{\mathcal{B}}^{-1}(x) = S\,x$$

$S = S_{\mathcal{B} \to \mathcal{U}}$ the j^{th} column of S is

$$S[e_j] = L_{\mathcal{U}}(L_{\mathcal{B}}^{-1}[e_j]) = L_{\mathcal{U}}(f_j) = [f_j]_{\mathcal{U}}$$

$$S = \begin{bmatrix} [f_1]_{\mathcal{U}} & [f_2]_{\mathcal{U}} & \cdots\cdots & [f_n]_{\mathcal{U}} \end{bmatrix}$$

<u>Example</u> $V = \operatorname{Span}\{e^x, e^{-x}\}$

$\sinh(x) = \dfrac{e^x - e^{-x}}{2}$ $\qquad \mathcal{B} = \{e^x, e^{-x}\}$

$\cosh(x) = \dfrac{e^x + e^{-x}}{2}$ $\qquad \mathcal{U} = \{\sinh x, \cosh x\}$

Compute $S_{\mathcal{B} \to \mathcal{U}} = S = \begin{bmatrix} [e^x]_{\mathcal{U}} & [e^{-x}]_{\mathcal{U}} \end{bmatrix}$

$[e^x]_{\mathcal{U}} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\qquad a \sinh x + b \cosh x = e^x$

$[e\ ]_{\mathcal{U}} = [b]$ $\quad a \sinh x + b \cosh x = e$

$x = 0 \Rightarrow b = 1$

take derivatives $\quad a \cosh x + \sinh x = e^x$

now set $x = 0 \Longrightarrow \quad a = 1$

$[e^{-x}]_{\mathcal{U}} = \begin{bmatrix} c \\ d \end{bmatrix}$ $\quad c \sinh x + d \cosh x = e^{-x}$

$x = 0 \quad d = 1$

$c \cosh x + d \sinh x = -e^{-x}$

$x = 0 \quad c = -1$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = S$$

<u>Theorem:</u> V subspace of $\mathbb{R}^n$. $\dim V = l \leq n$.

$\mathcal{U} = \{a_1, \dots, a_l\}$ basis of V

$\mathbb{B} = \{b_1, \dots, b_l\}$ basis of V

$$S = S_{\mathbb{B} \to \mathcal{U}}$$

$$S = \left[[b_1]_{\mathcal{U}} \ [b_2]_{\mathcal{U}} \cdots [b_l]_{\mathcal{U}} \right]$$

$$\begin{bmatrix} a_1 \cdots a_l \end{bmatrix} [b_1]_{\mathcal{U}} =$$

$$\begin{bmatrix} a_1 \cdots a_\ell \end{bmatrix} [b_1]_{\mathcal{U}} =$$

$$[X]_{\mathcal{U}} = \begin{bmatrix} c_1 \\ \vdots \\ c_\ell \end{bmatrix} \qquad \underbrace{c_1 a_1 + \cdots + c_\ell a_\ell}_{\begin{bmatrix} a_1\, a_2 \cdots a_\ell \end{bmatrix} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_\ell \end{bmatrix}}_{[X]_{\mathcal{U}}} = X} = X$$

$$\begin{bmatrix} a_1 \cdots a_\ell \end{bmatrix} [X]_{\mathcal{U}} = X$$ replace x by b_j

$$\begin{bmatrix} a_1 \cdots a_\ell \end{bmatrix} [b_j]_{\mathcal{U}} = b_j$$

$$\begin{bmatrix} a_1 \cdots a_\ell \end{bmatrix} \underbrace{\begin{bmatrix} [b_1]_{\mathcal{U}} \cdots [b_\ell]_{\mathcal{U}} \end{bmatrix}}_{S_{\mathcal{B} \to \mathcal{U}}} = \begin{bmatrix} b_1 \cdots b_\ell \end{bmatrix}$$

$$\begin{bmatrix} a_1 \cdots a_\ell \end{bmatrix} S_{\mathcal{B} \to \mathcal{U}} = \begin{bmatrix} b_1 \cdots b_\ell \end{bmatrix}$$

$\mathcal{U} = \{a_1, \ldots, a_\ell\}$ $\qquad$ $\mathcal{B} = \{b_1, \ldots, b_\ell\}$

$\mathcal{U} = \{a_1, \dots, a_\ell\}$ $\mathcal{B} = \{b_1, \dots, b_\ell\}$