$y'' + y = 0$

$y = C_1 \cos x + C_2 \sin x$

Def: A linear (or vector) space V is a set with two operation, an addition, $+$, and a scalar-vector multiplication, $\cdot$; such that the following is satisfied for all $f, g, h \in V$ and $\lambda, \beta \in \mathbb{R}$:

1) $(f+g)+h = f+(g+h)$
2) $f+g = g+f$
3) $\exists\, 0 \in V$ such that $f+0 = f$
4) $\exists\, -f \in V$ such that $f+(-f) = 0$
5) $\lambda(f+g) = \lambda f + \lambda g$
6) $(\lambda+\beta) f = \lambda f + \beta f$
7) $\lambda(\beta f) = (\lambda\beta) f$
8) $1 f = f$

Examples 1) $\mathbb{R}^n$

2) $F(\mathbb{R},\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \}$

$f, g \quad F(\mathbb{R},\mathbb{R}) \qquad \lambda \in \mathbb{R}$

$(f+g)(x) = f(x) + g(x)$

$(\lambda f)(x) = \lambda(f(x))$

3) $\mathbb{R}^{k \times n}$

Def: $f_1, f_2, \ldots, f_n \in V$. We say that f is a linear combination of $f_1, f_2, \ldots, f_n$ if $\exists\, c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that $f = c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$.

Example Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$. Let $B = \begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix}$.

Show that B is a linear combination of A & I.

$$B = c_1 I + c_2 A$$

$$\begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$2 = c_1$ $\quad 11 = c_1 + 3c_2$ $\quad c_1 = 2$ & $c_2 = 3$ ✓

$3 = c_2$

$6 = 2c_2$

Def: Subspaces. V vector space. $W \subset V$. W is subspace of V if:

1) $0 \in W$
2) If $f, g \in W$ then $f + g \in W$
3) If $f \in W$ and $\lambda \in \mathbb{R}$, then $\lambda f \in W$

Example $P = \{$set of polynomials$\}$

$a_0 + a_1 x + \cdots + a_n x^n \quad a_0, \ldots, a_n \in \mathbb{R}$

$p = a_0 + a_1 x + \cdots + a_n x^n$

$q = b_0 + b_1 x + \cdots + b_m x^m$

Let $k = \max\{n, m\}$ $\quad a_l = 0$ if $l > n$ $\quad b_l = 0$ if $l > m$

$p + q = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_k + b_k)x^k$

$\deg(p) = n$ if $a_n \neq 0$ and $a_l = 0$ for all $l > n$.

$P_n = \{ p$ polinomial: $\deg(p) \leq n$ or $p = 0\}$

P_n is a subspace.

1) $0 \in P_n$

2) $\deg(p+q) \leq \max\{\deg(p), \deg(q)\} \leq n$ if $p, q \in P_n$

3) $\deg(\lambda p) = \deg(p)$ if $\lambda \neq 0 \quad \lambda \in \mathbb{R}$

$\leq n$ if $p \in P_n$

Example: the solutions of $y'' + y = 0$ is a subspace of $F(\mathbb{R}, \mathbb{R})$ of $C^\infty(\mathbb{R})$ or $C^1(\mathbb{R})$

Def: $f_1, \ldots, f_n \in V$

1) $\mathrm{Span}\{f_1, \ldots, f_n\} = \{$all the linear combinations of $f_1, \ldots, f_n\}$

2) We say that $f_1, \ldots, f_n$ span V if $V = \mathrm{Span}\{f_1, \ldots, f_n\}$

3) We say that f_i is redundant if it is a linear combination of $f_1, f_2, \ldots, f_{i-1}$.

4) $f_1, \ldots, f_n$ are linearly independent if none of them is redundant.

5) $f_1, \ldots, f_n$ are linearly independent if and only if

$0 = c_1 f_1 + \cdots + c_n f_n$ implies $c_1 = \cdots = c_n = 0$

6) Let W be a subspace of V. We say $f_1, \ldots, f_n$ is a basis of W if they span W, and they are linearly independent. In this case, if $f \in W$ and

$f = c_1 f_1 + \cdots + c_n f_n$, the coefficients $c_1, \ldots, c_n$ are called the coordinates of f with respect to the basis $\mathcal{B} = (f_1, \ldots, f_n)$. The vector

$\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$ is called the $\mathcal{B}$-coordinate vector of f, and it is denoted by $[f]_{\mathcal{B}}$

$L(f) = [f]_{\mathcal{B}}$ $\quad L: V \to \mathbb{R}^n$ is called the $\mathcal{B}$-coordinate transformation, $L = L_{\mathcal{B}}$

Obs: $L: V \to \mathbb{R}^n$ $\quad L(f) = \begin{bmatrix} c_1 \\ \vdots \end{bmatrix} = [f]_{\mathcal{B}}$

Obs: $L: V \to \mathbb{R}^n$ $\quad L(f) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = [f]_{\mathcal{B}}$

$A: \mathbb{R}^n \to V \quad A\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1 f_1 + \cdots + c_n f_n$

A & L are inverses of each other

$A(L(f)) = A\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right) = c_1 f_1 + \cdots + c_n f_n = f$

$L\left(A\left(\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}\right)\right) = L(c_1 f_1 + \cdots + c_n f_n) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Obs:

$$\begin{aligned} & c_1 f_1 + \cdots + c_n f_n = f \\ + \; & d_1 f_1 + \cdots + d_n f_n = g \\ \hline & (c_1+d_1) f_1 + \cdots + (c_n + d_n) f_n = f+g \end{aligned}$$

$[f]_{\mathcal{B}} + [g]_{\mathcal{B}} = [f+g]_{\mathcal{B}}$ $\qquad \lambda \in \mathbb{R}$

$[\lambda f]_{\mathcal{B}} = \lambda [f]_{\mathcal{B}}$

Obs: Any two basis of the same subspace have the same number of elements

Def: $\dim(W) = \#$ of elements is a basis of W

Ex 1) $W = \{\text{solutions to } y'' + y = 0\}$

$W = \{ c_1 \sin x + c_2 \cos x \; : \; c_1, c_2 \in \mathbb{R} \}$

dim W? $\mathcal{B} = \{\sin x, \cos x\}$

If $y \in W$ then $y = c_1 \sin x + c_2 \cos x$ for some $c_1, c_2 \in \mathbb{R}$

then $y \in \text{Span}\{\sin x, \cos x\}$. then $W \subset \text{Span}\{\sin x, \cos x\}$.

But $\sin x \in W$, so does $\cos x$, then $\text{Span}\{\sin x, \cos x\} \subset W$

then $W = \text{Span}\{\sin x, \cos x\}$

Are $\sin x$, $\cos x$ linearly independent?

$$c_1 \sin x + c_2 \cos x = 0$$

Set $x = \frac{\pi}{2}$ $\quad c_1 \sin\frac{\pi}{2} + c_2 \cos\frac{\pi}{2} = 0$

$$c_1 = 0$$

Set $x = 0$ $\quad c_1 \sin(0) + c_2 \cos(0) = 0 \quad c_2 = 0$

then $\{\sin x, \cos x\}$ is linearly independent.

<u>Example</u>: Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$. Let $W = \{B \in \mathbb{R}^{2\times 2} : AB = BA\}$

1) Show W is a subspace of $\mathbb{R}^{2\times 2}$

2) Find a basis of W

1) a) $0 \in W$? $\quad 0A = 0 = A0 \Rightarrow 0 \in W$

b) B_1 & $B_2 \in W$ then

$AB_1 = B_1 A$

$$+\quad AB_2 = B_2 A$$

$$AB_1 + AB_2 = B_1 A + B_2 A$$

$$A(B_1 + B_2) = (B_1 + B_2) A \implies B_1 + B_2 \in W \checkmark$$

c) $B \in W \quad \lambda \in \mathbb{R} \implies AB = BA$

Multiply by λ $\quad \lambda(AB) = \lambda(BA)$

$$\underset{A(\lambda B)}{\parallel} \qquad \underset{(\lambda B)A}{\parallel} \implies \lambda B \in W$$

Finding a basis of W.

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W$

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

$$\parallel \qquad\qquad \parallel$$

$$\begin{bmatrix} c & d \\ 2a+3c & 2b+3d \end{bmatrix} = \begin{bmatrix} 2b & a+3b \\ 2d & c+3d \end{bmatrix}$$

$c = 2b \qquad\qquad 2b - c = 0$

$d = a+3b \qquad\qquad a+3b-d = 0$

$2a+3c = 2d \qquad\qquad 2a+3c-2d = 0$

$2b+3d = c+3d \qquad\qquad 2b - c = 0$

$$\begin{bmatrix} 0 & 2 & -1 & 0 \\ 1 & 3 & 0 & -1 \\ 2 & 0 & 3 & -2 \\ 0 & 2 & -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 1 & -1/2 & 0 \\ 0 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3/2 & -1 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$a = -\frac{3}{2}t_1 + t_2$

$b = \frac{1}{2}t_1$

$c = t_1$

$d = t_2$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -\frac{3}{2}t_1 + t_2 & \frac{1}{2}t_1 \\ t_1 & t_2 \end{bmatrix} = t_1 \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Basis of W $\quad \begin{bmatrix} -\frac{3}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$