1) $x_2 + x_3 - x_4 = 1$

$2x_1 - x_3 + 2x_4 = 2$

$-2x_1 + 2x_2 + 3x_3 - 4x_4 = 0$

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & -1 & 1 \\ 2 & 0 & -1 & 2 & 2 \\ -2 & 2 & 3 & -4 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -1/2 & 1 & 1 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 2 & 2 & -2 & 2 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & -1/2 & 1 & 1 \\ 0 & \boxed{1} & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\begin{aligned} x_1 &= 1 + \tfrac{1}{2}t_1 - t_2 \\ x_2 &= 1 - t_1 + t_2 \\ x_3 &= t_1 \\ x_4 &= t_2 \end{aligned}}$$

$$x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_1 \begin{bmatrix} 1/2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

2)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -2 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2$

3) $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\quad T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) =$

$\begin{bmatrix}3\\1\end{bmatrix} = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix}$ then

$$\boxed{T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = c_1 T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + c_2 T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)} = 2\begin{bmatrix}1\\0\\1\end{bmatrix} + (1)\begin{bmatrix}0\\2\\-1\end{bmatrix} = \boxed{\begin{bmatrix}2\\2\\1\end{bmatrix}}$$

$$\left[\begin{array}{cc|c}1 & 1 & 3\\1 & -1 & 1\end{array}\right] \quad \left[\begin{array}{cc|c}1 & 1 & 3\\0 & -2 & -2\end{array}\right] \quad \left[\begin{array}{cc|c}1 & 0 & 2\\0 & 1 & 1\end{array}\right]$$

$c_1 = 2 \quad c_2 = 1$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + \frac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) =$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}\right) = \frac{1}{2}T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - \frac{1}{2}T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$$

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\2\\-1\end{bmatrix} = \begin{bmatrix}1/2\\1\\0\end{bmatrix}$$

$$A = \begin{bmatrix}1/2 & 1/2\\1 & -1\\0 & 1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\2\\-1\end{bmatrix} = \begin{bmatrix}1/2\\-1\\1\end{bmatrix}$$

4) $L = \text{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$

$$\text{proj}_L(x) = \frac{\begin{bmatrix}2\\1\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}}{\begin{bmatrix}2\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\end{bmatrix}}\begin{bmatrix}2\\1\end{bmatrix} = \frac{2x_1 + x_2}{5}\begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix} \frac{4x_1+2x_2}{5} \\ \frac{2x_1+x_2}{5} \end{bmatrix} = \begin{bmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

5)
$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right]$$

$= A^{-1}$

6) $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & -1 & -1/2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & -1 & -1/2 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1/2 \end{bmatrix}$$

Basis of $\operatorname{Im}(A) = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$

$x_1 = -t/2$

$x_2 = -t/2 \qquad x = t \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

$x_3 = t$

Basis of $\ker(A) = \left\{ \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$

$\mathcal{B} = \{v_1, \dots, v_n\}$ basis of $\mathbb{R}^n$. $x \in \mathbb{R}^n$

$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ where $x = c_1 v_1 + \dots + c_n v_n$

$$x = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} [x]_{\mathcal{B}}$$

Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Assume $x = c_1 v_1 + \dots + c_n v_n$

$$\boxed{[T(x)]_{\mathcal{B}}} = [T(c_1 v_1 + \dots + c_n v_n)]_{\mathcal{B}} =$$

$$= [c_1 T(v_1) + \dots + c_n T(v_n)]_{\mathcal{B}} =$$

$$= c_1 [T(v_1)]_{\mathcal{B}} + c_2 [T(v_2)]_{\mathcal{B}} + \dots + c_n [T(v_n)]_{\mathcal{B}} =$$

$= c_1 [T(v_1)]_{\mathcal{B}} + c_2 [T(v_2)]_{\mathcal{B}} + \cdots + c_n [T(v_n)]_{\mathcal{B}} =$

$$= \boxed{\begin{bmatrix} [T(v_1)]_{\mathcal{B}} & [T(v_2)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}}$$

<u>Theorem</u> $[T(x)]_{\mathcal{B}} = \underbrace{\begin{bmatrix} [T(v_1)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} \end{bmatrix}} [x]_{\mathcal{B}}$

this matrix is called the $\mathcal{B}$-matrix of T

<u>Obs</u>: Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. $T(x) = Ax$. $[T(x)]_{\mathcal{B}} = B[x]_{\mathcal{B}}$

$$B = \begin{bmatrix} [T(v_1)]_{\mathcal{B}} & \cdots & [T(v_n)]_{\mathcal{B}} \end{bmatrix}$$

<u>Recall</u> $y \in \mathbb{R}^n$: $y = S[y]_{\mathcal{B}}$

$$S = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

$$S[T(x)]_{\mathcal{B}} = SB[x]_{\mathcal{B}}$$

$$\parallel$$

$$T(x) = Ax = AS[x]_{\mathcal{B}}$$

Then $\boxed{SB = AS}$ $\mathcal{B} = \{v_1, \dots, v_n\}$

$$S = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \qquad T(x) = Ax$$

$$[T(x)]_{\mathcal{B}} = B[x]_{\mathcal{B}}$$ $\boxed{\text{Theorem}}$

Ex. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$$

a) Find the matrix of T

b) " " $\mathcal{B}$-matrix of T

c) Verify that $SB = AS$ where $S = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

a) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

b) $B = \left[\left[T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\right]_{\mathcal{B}} \quad \left[T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)\right]_{\mathcal{B}} \right] = \begin{bmatrix} -5/3 & -1/3 \\ 7/3 & 5/3 \end{bmatrix}$

b) $B = \left[\left[T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)\right]_{\mathcal{B}} \quad \left[T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)\right]_{\mathcal{B}} \right] = \begin{bmatrix} & \\ 7/3 & 5/3 \end{bmatrix}$

$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix} = a_1\begin{bmatrix}1\\2\end{bmatrix} + a_2\begin{bmatrix}2\\1\end{bmatrix}$

$\left[\begin{array}{cc|c}1 & 2 & 3\\2 & 1 & -1\end{array}\right] \quad \left[\begin{array}{cc|c}1 & 2 & 3\\0 & -3 & -7\end{array}\right] \quad \left[\begin{array}{cc|c}1 & 0 & -5/3\\0 & 1 & 7/3\end{array}\right]$

$T\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$

$\left[\begin{array}{cc|c}1 & 2 & 3\\2 & 1 & 1\end{array}\right] \quad \left[\begin{array}{cc|c}1 & 2 & 3\\0 & -3 & -5\end{array}\right] \quad \left[\begin{array}{cc|c}1 & 0 & -1/3\\0 & 1 & 5/3\end{array}\right]$

$A = \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \quad B = \begin{bmatrix}-5/3 & -1/3\\7/3 & 5/3\end{bmatrix} \quad S = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}$

$SB = AS$

$\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}-5 & -1\\7 & 5\end{bmatrix}\frac{1}{3} = \begin{bmatrix}9 & 9\\-3 & 3\end{bmatrix}\frac{1}{3} = \begin{bmatrix}3 & 3\\-1 & 1\end{bmatrix}$

$\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} = \begin{bmatrix}3 & 3\\-1 & 1\end{bmatrix}$

Def. Let $A, B \in \mathbb{R}^{n\times n}$. We say that A is similar to B if there exists $S \in \mathbb{R}^{n\times n}$

invertible such that

$$SB = AS \quad \text{or} \quad B = S^{-1}AS \quad \text{or} \quad SBS^{-1} = A$$

Obs: 1) A similar to B and B similar to C implies A is similar to C.

$$A = S_1 B S_1^{-1} \qquad B = S_2 C S_2^{-1} \quad \text{then}$$

$$A = S_1 S_2 C S_2^{-1} S_1^{-1} = (S_1 S_2) C (S_1 S_2)^{-1}$$

2) A similar to B $\Rightarrow$ B similar to A

$$A = SBS^{-1} \qquad\qquad B = S^{-1} A (S^{-1})^{-1}$$

3) Any A is similar to itself

$$A = I A I^{-1}$$