**Def:** nullity of $A = \dim(\ker(A))$

**Reminder:** $S$ subspace. $\dim(S) = \#$ of elements in a basis of $S$.

**Example:** Find bases of the image and kernel of

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 1 & 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 1 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2t_1 - t_3$

$x_2 = t_1$

$x_3 = t_2$

$x_4 = -t_3$

$x_5 = t_3$

Basis of the image $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$

Basis of the kernel $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

**Example**

$$e_i = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow i^{th}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + x_2 e_2 + \cdots\cdots + x_n e_n$$

$$\operatorname{Span}\{e_1, \ldots, e_n\} = \mathbb{R}^n$$

$$0 = x_1 e_1 + \cdots + x_n e_n \Rightarrow x_1 = x_2 \cdots\cdots = x_n = 0$$

$e_1, \ldots, e_n$ are linearly independent.

Thus $e_1, \ldots, e_n$ is a basis of $\mathbb{R}^n$.

Obs: S subspace. $\dim(S) = r$. $v_1, \ldots, v_r \in S$, that are linearly independent. Then

$$W = \operatorname{Span}\{v_1, \ldots, v_r\} \subseteq S$$

$\dim W = r = \dim S$. Then $W = S$. Also, $v_1, \ldots, v_r$ is a basis of S.

Questions: $v_1, \ldots, v_r \subseteq \mathbb{R}^n$. Assume $v_1, \ldots, v_r$ are linearly independent.

1) How do $r$ & $n$ compare? $r \leq n$

2) If $r=n$, then $v_1, \dots, v_n$ is a basis of $\mathbb{R}^n$.

Summary

$A \in \mathbb{R}^{n \times n}$. The following statements are equivalent.

1) A is invertible
2) $Ax=b$ has a unique solution x for all $b \in \mathbb{R}^n$.
3) $\mathrm{rref}(A) = I$
4) $\mathrm{rank}(A) = n$
5) $\mathrm{Im}(A) = \mathbb{R}^n$
6) $\mathrm{Ker}(A) = \{0\}$
7) Span of columns of $A = \mathrm{l}^{\,n}$
8) The columns of A are a b · of $\mathbb{R}^n$
9) T columns of A are linearly in endent.

Done with section 3B

Obs/Def: Let S be a su space. Let $v_1, \dots, v_r$ be a basis of S. Let $x \in S$. l , there exists a unique set of coefficients $c_1, \dots, c_r$ such that

$$x = c_1 v_1 + c_2 v_2 + \dots + c_r v_r$$

$\mathfrak{B} = v_1, \dots, v_r$ we call this basis $\mathfrak{B}$

The numbers $c_1, \dots, c_r$ are called the $\mathfrak{B}$-coordinates of x. the vector $\begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}$ is called the $\mathfrak{B}$-coordinate vector of x and it is denoted by

B-coordinate vector of x and it is denoted by $[x]_B$. thus $[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix}$

$$\begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix} [x]_B = c_1 v_1 + \cdots + c_r v_r = x$$

<u>Example</u> $B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ Let $S = \text{Span}(B)$

Let

$x = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$. Q: Does $x \in S$? If yes, find the B-coordinates of x

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 1 & 2 & 7 \\ 1 & 3 & 9 \end{array}\right] \rightarrow \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{array}\right]$$

$c_1 = 3$

$c_2 = 2$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = [x]_B = \left[ \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} \right]_B$$

<u>Obs</u>: Given S subspace. B basis of S.

$\mathcal{B} = v_1, \ldots, v_r$. $x \in S$, then, the $\mathcal{B}$-coordinate of $x$, is the unique solution of

$$\underset{\substack{\uparrow \\ \text{given}}}{\begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix}} \; \underset{\substack{\uparrow \\ \text{solving} \\ \text{for this}}}{[x]_{\mathcal{B}}} = \underset{\substack{\uparrow \\ \text{given}}}{x}$$

**Obs:** $\mathcal{B}$ basis of $S$. $r = \dim S$

$T : S \longrightarrow \mathbb{R}^r$

$T(x) = [x]_{\mathcal{B}}$

$T$ is a linear transformation

proof: 1) $T(x+y) = T(x) + T(y)$

2) $T(\lambda x) = \lambda T(x)$

$T(x) = [x]_{\mathcal{B}}$ Let $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_r \end{bmatrix}$

$$\left.\begin{aligned} A[x]_{\mathcal{B}} &= x \\ A[y]_{\mathcal{B}} &= y \end{aligned}\right\} \Longrightarrow \begin{aligned} &A[x]_{\mathcal{B}} + A[y]_{\mathcal{B}} = x + y \\ &A(\underbrace{[x]_{\mathcal{B}}}_{} + \underbrace{[y]_{\mathcal{B}}}_{}) = x + y \end{aligned}$$

$A[y]_B = y$ ,     $A([x]_B + [y]_B) = x + y$

$A\underbrace{[x+y]_B} = x+y \longrightarrow [x+y]_B = [x]_B + [y]_B$

Also $[\lambda x]_B = \lambda [x]_B$

Example: $B = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$

Find $[x]_B$ where $x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

$$\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \underset{\begin{bmatrix}c_1\\c_2\end{bmatrix}}{\underset{\shortparallel}{[x]_B}} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$\left[\begin{array}{cc|c}1 & 1 & x_1\\1 & -1 & x_2\end{array}\right]$$

$$\left[\begin{array}{cc|c}1 & 1 & x_1\\0 & -2 & x_2 - x_1\end{array}\right] \longrightarrow \left[\begin{array}{cc|c}1 & 0 & (x_1+x_2)/2\\0 & 1 & \frac{x_1 - x_2}{2}\end{array}\right]$$

$$[x]_B = \begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}\frac{x_1+x_2}{2}\\ \frac{x_1-x_2}{2}\end{bmatrix}$$

Check $\frac{x_1+x_2}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{x_1-x_2}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$