MATH 4305

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A. Section 3.2

Definition: $v_1, \ldots, v_2 \in \mathbb{R}^n$

1) v_i is *redundant* if v_i is a linear combination of $v_1, \ldots, v_{i-1}$.

Let $A \in \mathbb{R}^{k \times n}$,

$$A = \begin{bmatrix} a_1 & a_2 & \ldots & a_3 \end{bmatrix}$$

Question: What columns of A are redundant?

Answer: The non-pivot columns are the redundant ones.

Compute the rref(A). The columns that do not have leading ones, are the redundant columns (correspond to free variables). A basis of the Im(A) is obtained by keeping the non-redundant columns of A.

Example: Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}.$$

Find the redundant columns of A and find a basis of Im(A).

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

First and second are the *pivot* columns. Third column is *redundant*.

$$\text{Basis of Im}(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Example:

$$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Find a basis of S. This is the same as finding a basis of

$$\text{Im}\left\{ \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \right\}$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis of S is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\begin{bmatrix} 2 & 2 & 0 & 0 \end{bmatrix}^T$ is redundant.

Let $A \in \mathbb{R}^{k \times n}$. How do we find a basis of $\ker(A)$.

$\ker(A) = \{x \in \mathbb{R}^n : Ax = 0\}$ (All the solutions of $Ax = 0$)

Do Gaussian Elimination to get that any $x \in \ker(A)$ is of the form

$$x = t_1\vec{v}_1 + t_2\vec{v}_2 + \cdots + t_r\vec{v}_r$$

These vectors $v_1, v_2, \ldots, v_r$ that span the kernel have to be linearly independent. The form the basis if $\ker(A)$.

Example:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 2 & 1 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis of the image are the first two columns.

$$\begin{cases} x_1 &= -2t_1 - 2t_2 \\ x_2 &= 2t_1 + t_2 \\ x_3 &= t_1 \\ x_4 &= t_2 \end{cases}, \quad x = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} t_1 + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} t_2$$

$$\text{Basis of } \ker(A) = \left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Example: Find a basis of the solution of the system (set of solutions forms the subspace, RHS has to be zero - otherwise zero vector is not a solution, thus, not a subspace):

$$x_1 - x_2 + x_3 = 0$$
$$x_1 - x_3 + x_4 = 0$$
$$2x_1 - x_2 + x_4 = 0$$

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 2 & -1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{cases} x_1 & = t_1 - t_2 \\ x_2 & = 2t_1 - t_2 \\ x_3 & = t_1 \\ x_4 & = t_2 \end{cases}$$

$$x = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} t_1 + \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} t_2$$

$$\text{Basis} s = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Definition: $v_1, v_2, \ldots, v_r \in \mathbb{R}^n$. An equation of the form

$$\lambda_1 v_1 + \cdots + \lambda_r v_r = 0$$

is called a relation among the vectors $v_1, \ldots, v_r$. The relation is *non-trivial* if at least one of the $\lambda' s \neq 0$.

Observation: $v_1, \ldots, v_r$ is linearly dependent if there is a relation that is *non-trivial*.

Observation: $v_1, \ldots, v_r \in \mathbb{R}^n$. The following are equivalent:

1) The vectors are linearly independent.
2) None of the vectors are redundant.
3) None of the v_i is a linear combination of the other vectors.
4) There are no non-trivial relations among these vectors.
5) $\ker \begin{bmatrix} v_1, \ldots, v_r \end{bmatrix} = \{0\}$
6) $\text{rank} \begin{bmatrix} v_1, \ldots, v_r \end{bmatrix} = r$

Observation: Let $S = \text{span}\{v_1, \ldots, v_r\}$. Let $x \in S$.

$$x = \lambda_1 v_1 + \cdots + \lambda_r v_r, \text{ for some } \lambda_1, \ldots, \lambda_r \in \mathbb{R}$$

Claim: the set of numbers $\lambda_1, \ldots, \lambda_r$ is unique if and only if $v_1, \ldots, v_r$ is a basis of S.

Proof: We know that

$$\begin{aligned} c_1 v_1 + \cdots + c_r v_r &= 0 \\ 0 v_1 + \cdots + 0 v_r &= 0 \end{aligned}$$

These are two linear combinations that give $\vec{0}$. But we are assuming that there is only one set of coefficients for each $x \in S$, in particular for $x = 0$. Then, $c_1 = 0, c_2 = 0, \ldots, c_r = 0$.

$$\begin{aligned} x &= c_1 v_1 + \cdots + c_r v_r \\ x &= d_1 v_1 + \cdots + d_r v_r \\ 0 &= (c_1 - d_1) v_1 + \cdots + (c_r - d_r) v_r \end{aligned}$$

Since $v_1, \ldots v_r$ are linearly independent, then $c_1 = d_1, c_2 = d_2, \ldots c_r = d_r$. If $x = c_1 v_1 + \cdots + c_r v_r$ and $v_1, \ldots, v_r$ are linearly independent, the coefficients are called the *coordinates* of x with respect to the basis $v_1, \ldots, v_r$.

Finished Section 3.2.

B. Dimension of a subspace

Let S be a subspace. The dimension of S is $\dim S =$ number of vectors in a basis of S.

Observation: two different basis of the same subspace have the same number of elements.

Observation: Let $A \in \mathbb{R}^{k \times n}$.

$$\begin{aligned} \dim(\text{Im(A)}) &= \text{\# of pivot columns of } A \\ &= \text{\# of leading ones in rref}(A) \\ &= \text{rank}(A) \\ n &= \text{\# of leading variables} + \text{\# of free variables} \end{aligned}$$

$$\boxed{\text{rank}(A) + \dim(\ker(A))} = n$$

The sum of the dimension of A and the dimension of the kernel of A = n = number of columns of A