Week 5

1 Week 5

1.1 Lecture 1

Example. Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$

Solution: We do

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{array}\right]$$

We conclude that

$$A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$$

Indeed, it can be verified that $AA^{-1} = A^{-1}A = I$.

- OBS: A and $B \in \mathbb{R}^{n\times n}$ both invertible. Then AB is also invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.
  Proof. Note that

  $$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

  Similarly,

  $$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$$

  proving the claim.

- Fact: $A, B \in \mathbb{R}^{n\times n}$ such that $BA = I$. Then
  (1) Both A and B are invertible.
  (2) $A^{-1} = B$
  (3) $AB = I$

- OBS:

  $$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & -bc+da \end{bmatrix} = (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition: The determinant of a 2×2 matrix $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is

$$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$$

- Fact: Let $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Then A is invertible if and only if $\det A \neq 0$. In this case we have

  $$\det A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

  Example: $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$. Note that then we have $\det A = \frac{1}{3}$, and therefore

  $$A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{bmatrix}$$

- OBS: The linear transformation corresponding to the rotation by θ counter-clockwise has matrix

  $$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

  If $\theta = \frac{\pi}{2}$ then $R_{\pi/2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
  Observe that

  $$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc = \begin{bmatrix} -b \\ a \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \left|\left| \begin{bmatrix} -b \\ a \end{bmatrix} \right|\right| \times \left|\left| \begin{bmatrix} c \\ d \end{bmatrix} \right|\right| \cos\left(\frac{\pi}{2} - \theta\right)$$

  $$= \sqrt{a^2+b^2}\sqrt{c^2+d^2}\sin\theta$$

- Fact: Let $v, w \in \mathbb{R}^2$. Then

  $$\left| \det \begin{bmatrix} v & w \end{bmatrix} \right| = \text{Area of a parallelogram}$$

END OF CHAPTER 2

Definition. Let $f : X \to Y$. The image of f is

$$Im(f) \equiv \{y \in Y : \exists x \in X \text{ s.t. } y = f(x)\}$$

Example. $f(x) = x^2, f : \mathbb{R} \to \mathbb{R}$. Then $Im(f) = \{y \in \mathbb{R} : y \geq 0\}$.

Definition. Let $v_1, v_2, \ldots, v_r \in \mathbb{R}^n$. Then we define

$$\text{span}\{v_1, v_2, \ldots, v_r\} = \{\text{set of all linear combinations of } v_1, v_2, \ldots, v_r\}$$

Recall that x is a linear combination of $v_1, v_2, \ldots, v_r$ is $\exists \lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{R}$ such that

$$x = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_r v_r$$

Example. Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Then $x = 3v_1 + 2v_2 = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$ is a linear combination of v_1, v_2.

Example. Let $v \in \mathbb{R}^n$. Then

$$\text{span}\{v\} = \{x \in \mathbb{R}^n : x = \lambda v, \text{ for some } \lambda \in \mathbb{R}\} = \{\lambda v : \lambda \in \mathbb{R}\}$$

- OBS: If v_1 is not a multiple of v_2 and v_2 is not a multiple of v_1, then $\text{span}\{v_1, v_2\}$ is the plane that contains v_1, v_2 and the origin.

Let $T : \mathbb{R}^n \to \mathbb{R}^k$ be a linear transformation. Let $A \in \mathbb{R}^{k \times n}$ be its matrix, i.e., $T(x) = Ax, \forall x \in \mathbb{R}^n$. Then

$$Im(f) = \{y \in \mathbb{R}^k : y = T(x)\} = \{T(x) : x \in \mathbb{R}^n\} = \{Ax : x \in \mathbb{R}^n\}$$

Let $A = \begin{bmatrix} A_1 & A_2 & \ldots & A_n \end{bmatrix}$, with A_j being the jth column of A, and let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Then

$$Im(f) = \left\{ \begin{bmatrix} A_1 & A_2 & \ldots & A_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \right\} = \left\{ \sum_{j=1}^{n} x_j A_j : x_j \in \mathbb{R}, 1 \le j \le n \right\}$$
$$= \text{span}\{A_1, A_2, \ldots, A_n\}$$

- OBS: $T(x) = Ax$. Then $Im(T) = \text{span}\{\text{columns of } A\}$

Definition. $Im(A) = \text{span}\{\text{columns of } A\}$.

Example. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. Let $v = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$. Does $v \in Im(A)$?

To answer this question, we attempt to solve the linear system

$$v = x_1 A_1 + x_2 A_2 = x_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Applying Gaussian elimination, we obtain the augmented matrix

$$\left[\begin{array}{cc|c} 2 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & -1 \\ 0 & 1/2 & 3/2 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & -1 \\ 0 & 1/2 & 3/2 \\ 0 & 0 & -2 \end{array}\right]$$

implying $0 = -12$, which implies that $\nexists x_1, x_2 \in \mathbb{R} : v = x_1 A_1 + x_2 A_2$. It follows that $v \notin Im(A)$.

- $b = Ax$ is consistent $\iff$ $b \in \text{span}\{\text{columns of } A\} = Im(A)$, i.e., $Ax = b$ has at least one solution $\iff$ $b \in Im(A)$.

Definition:

(1) Let $T : \mathbb{R}^n \to \mathbb{R}^k$ be a linear transformation. The *Kernel* of T is defined as

$$ker(T) = \{x \in \mathbb{R}^n : T(x) = 0\}$$

(2) Let $A \in \mathbb{R}^{k \times n}$. Then $ker(A) = \{x \in \mathbb{R}^n : Ax = 0\}$.

- OBS: $0 \in ker(T)$, or equivalently, $0 \in ker(A)$.

  Example. Find the kernel of $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. We solve the system $Ax = 0$. Using row operations we have

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

  Then we set $x_1 = -t$, $x_2 = -t$ and $x_3 = t$, for any $t \in \mathbb{R}$. It follows that

$$x = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \tag{1}$$

  Therefore, as any solution x to the system $Ax = 0$ is of the form in (1), we conclude that

$$ker(A) = \text{span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

1.2 Lecture 2

Summary of lecture 1:

- $ker(A) = \{x : Ax = 0\} \subset \mathbb{R}^n$.
- $Im(A) = \{y : \exists x : y = Ax\} \subset \mathbb{R}^k$.
- $A \in \mathbb{R}^{k \times n}$. $T(x) = Ax$. Then $ker(T) = ker(A)$, and $Im(T) = Im(A)$.

Facts:

- $A \in \mathbb{R}^{k \times n}$. Then $ker(A) = \{0\} \iff \text{rank}(A) = n$.
- If $ker(A) = \{0\}$, then $\text{rank}(A) = n \le k$.
- Let $k = n$. Then $ker(A) = \{0\} \iff A$ is invertible.

In sumary: Let $A \in \mathbb{R}^{n \times n}$. The following statements are equivalent:

- A is invertible.
- The linear system $Ax = b$ has a unique solution, $\forall b$.
- $\text{rref}(A) = I$.
- $\text{rank}(A) = n$.
- $Im(A) = \mathbb{R}^n$.
- $ker(A) = \{0\}$.

Definition. Let $S \subset \mathbb{R}^n$. S is called *a subspace of* $\mathbb{R}^n$ if:

(1) $0 \in S$

(2) $v, w \in S \implies v + w \in S$

(3) $v \in S, \lambda \in \mathbb{R} \implies \lambda v \in S$

Examples.

(1) $S = \{0\}$ is a subspace.

(2) $S = \mathbb{R}^n$ is a subspace.

(3) Let $v \in \mathbb{R}^n$. The smallest subspace that contains v is the line $L = \{\lambda v : \lambda \in \mathbb{R}\}$. Indeed,

  (a) $0 \in L$, since $0 = 0 \times v$.

  (b) Let $w_1, w_2 \in L$. Then $w_1 = \lambda_1 v, w_2 = \lambda_2 v$, and therefore $w_1 + w_2 = (\lambda_1 + \lambda_2)v \in L$, since $\lambda_1 + \lambda_2 \in \mathbb{R}$.

  (c) Let $w \in L$ and $\beta \in \mathbb{R}$, then $w = \lambda v$, and therefore $\beta w = (\beta\lambda)v \in L$, since $\beta\lambda \in \mathbb{R}$.

(4) Let $v_1, v_2 \in \mathbb{R}^n$. Then the smallest subspace of $\mathbb{R}^n$ that contains v_1 and v_2 is the plane $P = \{\text{plane that contains } v_1, v_2 \text{ and } 0\}$.

(5) $S = \{(x, y) : x^2 + y^2 = 1\} \subset \mathbb{R}^2$ *is not* a subspace.

(6) $S = \{(x, y) : y = x^2\} \subset \mathbb{R}^2$ *is not* a subspace.

Theorem: Let $A \in \mathbb{R}^{k \times n}$. Then $ker(A)$ is a subspace of $\mathbb{R}^n$, and $Im(A)$ is a subspace of $\mathbb{R}^k$.
Proof. It suffices to verify the definition of subspace. Indeed,

(1) $A \times 0 = 0$, and thus $0 \in ker(A)$.

(2) Let $v_1, v_2 \in ker(A)$. Then $Av_1 = Av_2 = 0$, and therefore $A(v_1 + v_2) = Av_1 + Av_2 = 0 \implies v_1 + v_2 \in ker(A)$.

(3) Let $v \in ker(A)$ and $\lambda \in \mathbb{R}$. Then $A(\lambda v) = \lambda(Av) = \lambda \times 0 = 0$. Then $\lambda v \in ker(A)$.

We conclude that $ker(A)$ is a subspace of $\mathbb{R}^n$. Similarly for $Im(A)$,

(1) As $0 \in \mathbb{R}^n$ is such that $A \times 0 = 0$, we have that $0 \in Im(A)$.

(2) Let $w_1, w_2 \in Im(A)$. Then $\exists v_1, v_2 \in \mathbb{R}^n : w_1 = Av_1$ and $w_2 = Av_2$. It follows that $w_1 + w_2 = Av_1 + Av_2 = A(v_1 + v_2)$. Since $v_1 + v_2 \in \mathbb{R}^n$, then $\exists v = v_1 + v_2 \in \mathbb{R}^n : w_1 + w_2 = Av$, implying that $w_1 + w_2 \in Im(A)$.

(3) Let $w \in Im(A)$ and $\lambda \in \mathbb{R}$. Then $\exists v \in \mathbb{R}^n : w = Av$. It follows that $\lambda w = \lambda(Av) = A(\lambda v)$. As $\lambda v \in \mathbb{R}^n$, then $\exists v' = \lambda v \in \mathbb{R}^n : \lambda w = Av' = A(\lambda v)$, following that $\lambda w \in Im(A)$.

concluding the proof.

Recall: Let $v_1, \dots, v_n \in \mathbb{R}^k$. Then $\text{span}\{v_1, \dots, v_n\} = \{\sum_{i=1}^n \lambda_i v_i : \lambda_i \in \mathbb{R}\}$ is a subspace, since $\text{span}\{v_1, \dots, v_n\} = \{\sum_{i=1}^n \lambda_i v_i : \lambda_i \in \mathbb{R}\} = Im(\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix})$.

Definition. Let $v_1, \dots, v_r \in \mathbb{R}^n$. The set $\{v_1, \dots, v_r\}$ is linearly independent (or equivalently, the vectors $v_1, \dots, v_r$ are linearly independent) if

$$x_1 v_1 + \cdots + x_r v_r = 0 \implies x_1 = x_2 = \cdots = x_r = 0$$

OBS:

- Note that

$$\begin{bmatrix} v_1 & \dots & v_r \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} = x_1 v_1 + \dots x_r v_r$$

  Then $v_1, \dots, v_r$ are linearly independent if and only if $ker(\begin{bmatrix} v_1 & \dots v_r \end{bmatrix}) = \{0\}$.

- Assume $v_1, \dots, v_r$ is linearly dependent (not linearly independent). Then we can have $\sum_{i=1}^r x_i v_i = 0$ and $x_k \neq 0$, for some $k \in \{1, \dots, r\}$ (can be multiple indices k). Let l be the largest of them (i.e., l is such that $x_l \neq 0$ and $x_i = 0, \forall i \in \{l+1, \dots, r\}$. Then

$$\sum_{i=1}^r x_i v_i = 0 \implies \sum_{i=1}^l x_i v_i = 0 \implies v_l = -\frac{1}{x_l} \sum_{i=1}^{l-1} x_i v_i$$

- $v_1, \dots, v_r$ are linearly dependent if and only if there exists $l \in \{1, 2, \dots, r\}$ such that v_l is a linear combination of $v_1, v_2, \dots v_{l-1}$.

- Let $v \in \mathbb{R}^n$. When is v linearly independent? Note that for $x \in \mathbb{R}^n$, $xv = 0 \iff x = 0$ or $v = 0$. Thus v is linearly independent if and only if $v \neq 0$.

- Let $v_1, v_2 \in \mathbb{R}^n$, with $v_1 \neq 0$, $v_2 \neq 0$ (if one of them is 0, then any set of vectors containing that vector is linearly dependent).
  v_1, v_2 are linearly dependent if and only if one of them is 0, or v_2 is a multiple of v_1, i.e., $v_2 = \lambda v_1$, for some $\lambda \in \mathbb{R}$.

Definition. Let $S \subset \mathbb{R}^n$ be a subspace. We say the set $\{v_1, v_2, \dots, v_r\}$ is a basis of S if:

(1) $S = \text{span}\{v_1, \dots, v_r\}$.

(2) $\{v_1, \dots, v_r\}$ is linearly independent.

Given a subspace S, we would like to obtain a basis of such S. it can be noted that we can be given a subspace S in terms of the kernel (as a set of solutions to an homogeneous linear system of equations), or as the image of a matrix (if we are given a set of vectors that span S).

Examples.

(1) S solutions to

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ x_1 + x_3 &= 0 \end{aligned}$$

thus $S = ker\left(\begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}\right)$.

(2) Let $S = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}\right\}$. Then $S = Im\left(\begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 5 \end{bmatrix}\right)$.