Product of matrices

Def: $A \in \mathbb{R}^{p\times n}$, $B \in \mathbb{R}^{k\times p}$. Let

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^p \quad T(x) = Ax$

$S : \mathbb{R}^p \longrightarrow \mathbb{R}^k \quad S(y) = By$

BA is the matrix such that

$(S \circ T)(x) = (BA)\, x$

Obs: BA is defined if the number of columns of B equcels the number of rows of A.

If $A \in \mathbb{R}^{p\times n}$ then, if $x \in \mathbb{R}^n$, $Ax \in \mathbb{R}^p$

If we can do $B(Ax)$ and $Ax \in \mathbb{R}^p$, then $B \in \mathbb{R}^{k\times p}$

$$\underset{\mathbb{R}^{k\times p}}{\overset{\in}{B}} \ \underset{\mathbb{R}^{p\times n}}{\overset{\in}{A}}$$

Obs. the j^{th} column of $(BA)e_j$ $\quad e_j = \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow j^{th} \\ \\ \\ \\ \end{array}$

__Obs__: the j^{th} column of $(BA)e_j$ $\quad e_j = \begin{bmatrix} \vdots \\ 0 \end{bmatrix}$

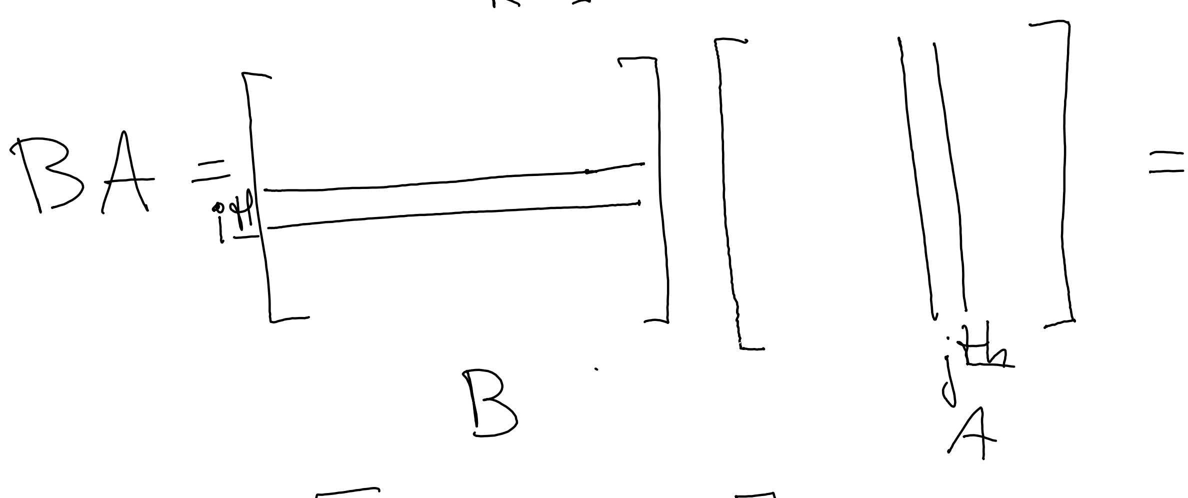
$$(BA)e_j = B(Ae_j) = Ba_j$$

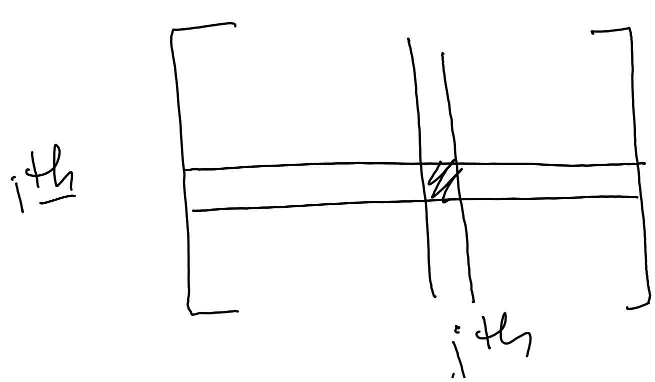
a_j is the j^{th} column of A

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$$BA = \begin{bmatrix} Ba_1 & Ba_2 & \cdots & Ba_n \end{bmatrix}$$

__Obs__: $B = \begin{bmatrix} b_1^T \\ \vdots \\ b_k^T \end{bmatrix}$ $\quad x = \begin{bmatrix} b_1^T x \\ b_2^T x \\ \vdots \\ b_k^T x \end{bmatrix} = \begin{bmatrix} b_1 \cdot x \\ b_2 \cdot x \\ \vdots \\ b_k \cdot x \end{bmatrix}$





jth

Ex: $\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2(1)+(-1)0+0(-1) \\ 1(1)+0(0)+3(-1) \end{bmatrix}$

$\begin{matrix} 6 & -1 \\ 2 & 4 \end{bmatrix} =$

$$= \begin{bmatrix} 2 & 6 & -1 \\ -2 & 2 & 4 \end{bmatrix}$$

Obs Multiplication of matrices is not commutative, in general $AB \neq BA$

Obs: $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \in \mathbb{R}^{n\times n}$

$AI = \begin{bmatrix} Ae_1 & Ae_2 & \cdots & Ae_n \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} = A$

$Ix = \begin{bmatrix} e_1^T \\ e_2^T \\ \vdots \\ e_n^T \end{bmatrix} x = \begin{bmatrix} e_1 \cdot x \\ e_2 \cdot x \\ \vdots \\ e_n \cdot x \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x$

$I \in \mathbb{R}^{n\times n}$ $\quad x \in \mathbb{R}^n$

$\cdots \,] = A$

$I \in \mathbb{R}^{n\times n}$ $x \in \mathbb{R}^n$

$$IA = [Ia_1 \; Ia_2 \ldots Ia_n] = [a_1 \; a_2 \ldots a_n] = A$$

<u>Obs</u>: $(AB)C = A(BC)$? yes

$Ax = T(x)$

$By = S(y)$

$Cz = R(z)$

$$((AB)C)z = ((T\circ S)\circ R)(z) = (T\circ S)(R(z)) = T(S(R(z))) = T((S\circ R)(z)) = (T\circ(S\circ R))(z) = (A(BC))z$$

thus $(AB)C = A(BC)$.

<u>Obs</u>: $A(C+D) = AC + AD$

$(A+B)C = AC + BC$

$(\lambda A)B = A(\lambda B) = \lambda(AB)$

We finished 2.3

Inverse of linear transformations and

matrices

<u>Def</u>: $f: X \to Y$. f has an inverse if for all $y \in Y$ there exists a unique $x \in X$ such that $f(x) = y$. the inverse of f is denoted by f^{-1}. $f^{-1}: Y \to X$

$f^{-1}(y) = x$ if and only if $f(x) = y$

<u>Obs</u>: $f(f^{-1}(y)) = y$ $\qquad f^{-1}(f(x)) = x$

$(f^{-1})^{-1} = f$

<u>Obs</u>: $T(x) = Ax$ $\qquad A \in \mathbb{R}^{k \times n}$

When is T invertible? (this means it has an inverse)

$Ax = b$ needs to have only one solution for all b

$[A \mid b]$ there are no free variables

rank $A = n \leq k$

later we will show that T^{-1} is also a linear transformation $T^{-1}: \mathbb{R}^k \to \mathbb{R}^n$

then $k \leq n$.

<u>Obs</u>: Let $A \in \mathbb{R}^{n \times n}$. The following are equivalent:

1) $T(x) = Ax$ is invertible
2) $\operatorname{rank}(A) = n$
3) $\operatorname{rref}(A) = I$
4) $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$
5) $Ax = 0$ has $x = 0$ as the unique solution

<u>Obs</u>: $T(x) = Ax$ invertible. Then $T^{-1}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is also a linear trans-

formation.

1) Let $x, y \in \mathbb{R}^n$

$T^{-1}(x+y) = v \Rightarrow T(v) = x+y$

$T^{-1}(x) + T^{-1}(y) = w \Rightarrow T(w) = T(T^{-1}(x) + T^{-1}(y)) =$

$= T(T^{-1}(x)) + T(T^{-1}(y)) = x+y$

then $T(v) = T(w)$, since T is invertible $\Rightarrow$

T is one to one, thus $v = w$, thus

$$T^{-1}(x+y) = T^{-1}(x) + T^{-1}(y)$$

Similarly, we can show that $T^{-1}(\lambda x) = \lambda T^{-1}(x)$.

thus, T^{-1} is a linear transformation.

$$T^{-1}(x) = B\,x \quad \text{for some } B \in \mathbb{R}^{n \times n}$$

this B is called the inverse of A, where $T(x) = Ax$

and B is denoted by A^{-1}

<u>Obs</u>: $(T^{-1} \circ T)(x) = x$

$$(A^{-1}A)\, x = I x$$

thus $A^{-1}A = I$. Similarly, $AA^{-1} = I$

Def: $A \in \mathbb{R}^{n\times n}$. 1) A is invertible if $T(x) = Ax$ is invertible.

2) The inverse of A is A^{-1}. $T^{-1}(x) = A^{-1}x$.

Obs: Let $A \in \mathbb{R}^{n\times n}$. A is invertible $\Longleftrightarrow$ there exist $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I$.

In this case, this $B = A^{-1}$

Computing A^{-1}:

Let $B = A^{-1} = [b_1\ b_2 \cdots b_n]$

$$AB = I$$

$$A b_j = e_j$$

$[A \mid e_1 e_2 \cdots e_n]$ do Gaussian elimination to the R.R.E.F of this augmented matrix

Case 1. No free variables rank $A = n$. The RREF

is $[I \mid b_1 b_2 \dots b_n]$

then $A^{-1} = [b_1 b_2 \dots b_n]$

Case 2. Some free variables, then rank $A < n$,

then A does not have an inverse.