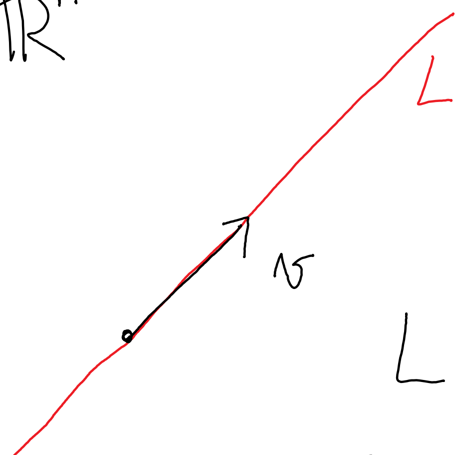
Lines through the origin

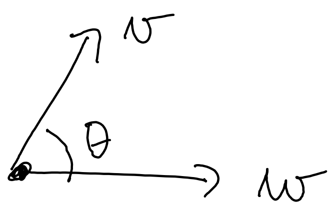
$v \in \mathbb{R}^n$



L is the line through the origin that contains the vector v (L is in the direction of v).

$$L = \{\lambda v : \lambda \in \mathbb{R}\}$$

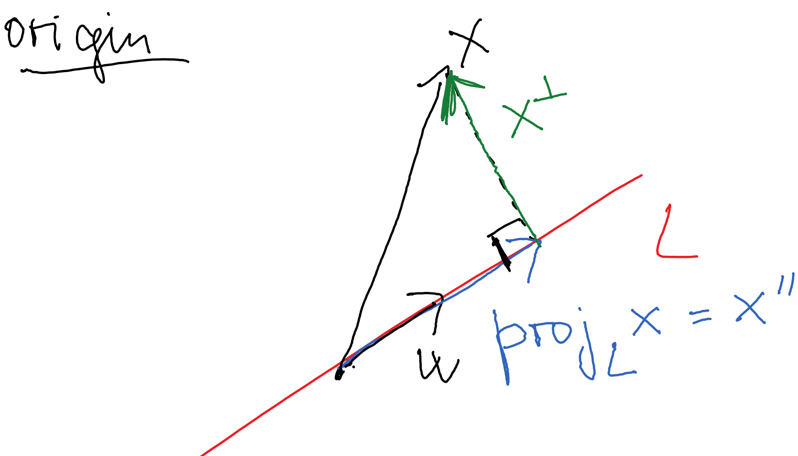
Recall:



$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|}$$

In particular $v \perp w$ (this means v is orthogonal or perpendicular to w) if $v \cdot w = 0$.

Orthogonal projection onto a line through the origin



$$x = x^{\parallel} + x^{\perp}$$

$$w \cdot x = w \cdot x^{\parallel} + w \cdot x^{\perp},$$

$$\underbrace{}_{= 0}$$

$x^{\parallel} = \lambda w$

$$w \cdot x = w \cdot (\lambda w) = \lambda (w \cdot w)$$

$$\lambda = \frac{w \cdot x}{w \cdot w} \qquad \boxed{\mathrm{proj}_L x = \left(\frac{w \cdot x}{w \cdot w}\right) w}$$

$$x^{\perp} = x - \mathrm{proj}_L x$$

<u>Obs</u>: If $L = \{\lambda w : \lambda \in \mathbb{R}\} \subset \mathbb{R}^n$ then $\mathrm{proj}_L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. this is a linear transformation

$$\mathrm{proj}_L x = \left(\frac{w \cdot x}{w \cdot w}\right) w = A x$$

what is A?

$$A = \frac{1}{(w \cdot w)} \begin{bmatrix} w_1 w^T \\ w_2 w^T \\ \vdots \\ w_n w^T \end{bmatrix}$$

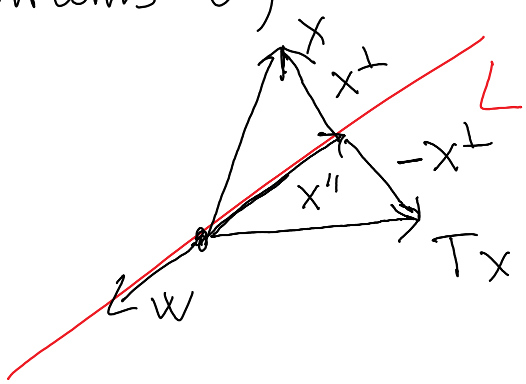
<u>Example</u>: $w = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad L = \lambda w$

$$\mathrm{proj}_L x = \left(\frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}{}\right) \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{x_1 + 2x_2}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{proj}_L x = \left(\frac{\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}x_1\\x_2\end{bmatrix}}{\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\2\end{bmatrix}} \right)\begin{bmatrix}1\\2\end{bmatrix} = \frac{x_1+2x_2}{5}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1/5 & 2/5\\ 2/5 & 4/5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$A = \frac{1}{(w\cdot w)}\begin{bmatrix} w_1\, w^T \\ w_2\, w^T \end{bmatrix} = \frac{1}{w\cdot w}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}$$

__Reflections__ (over a Line in $\mathbb{R}^2$, the line contains 0)



$$Tx = x^{\parallel} - x^{\perp}$$

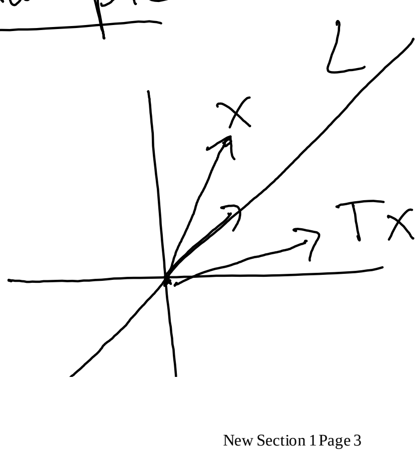
$$x^{\perp} = x - \text{proj}_L x$$

$$x^{\parallel} = \text{proj}_L x$$

$$Tx = 2\,\text{proj}_L x - x$$

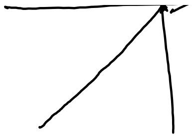
$$Tx = \left(\frac{2}{w\cdot w}\begin{bmatrix} w_1\, w^T \\ \vdots \\ w_n\, w^T \end{bmatrix} - I \right) x$$

__Example__



$$w = \begin{bmatrix}1\\1\end{bmatrix} \qquad Tx = \begin{bmatrix}2\\1\end{bmatrix}$$

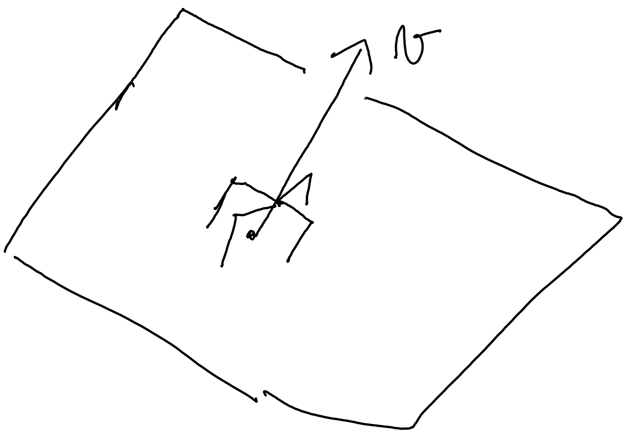
$$x = \begin{bmatrix}1\\2\end{bmatrix}$$



$$Tx = 2\,\mathrm{proj}_L x - x = \frac{2}{w\cdot w}(w\cdot x)\,w - x =$$

$$= \frac{2}{2}(x_1+x_2)\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_2\\x_1\end{bmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

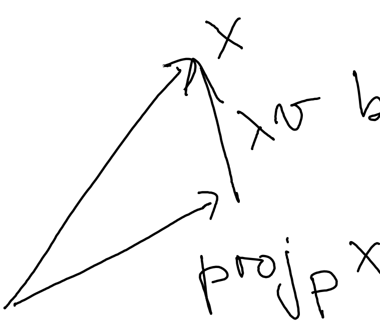
Planes in $\mathbb{R}^3$ containing 0



Given $v \in \mathbb{R}^3$, all the vectors perpendicular to v define a plane that contains the origin

$$x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$$

<u>Projection of x onto a plane P: $v\cdot x = 0$</u>



λv because this vector is $\perp$ to P

$x - \mathrm{proj}_P x$ is $\perp$ to P

$x - \mathrm{proj}_P x = \lambda v$ · dot product with v

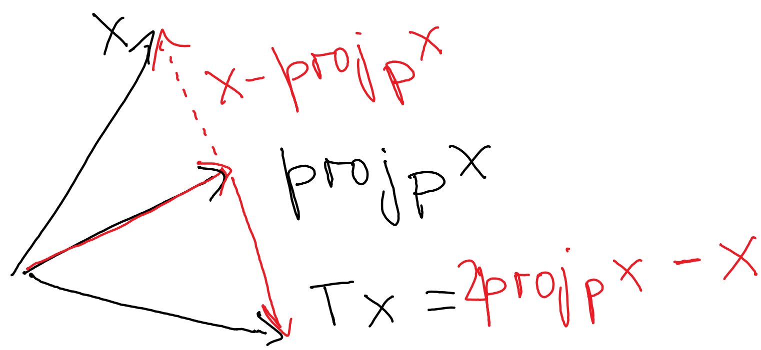
$v \cdot x - \underbrace{v \cdot \mathrm{proj}_P x}_{=0} = \lambda (v \cdot v)$

$=0$ because $\mathrm{proj}_P x \in P$ and the plane is $v \cdot z = 0$ (all vector $z \in \mathbb{R}^3$ such that $v \cdot z = 0$).

$\lambda = \dfrac{v \cdot x}{v \cdot v}$ then $\mathrm{proj}_P x = x - \left(\dfrac{v \cdot x}{v \cdot v}\right) v$

Reflection

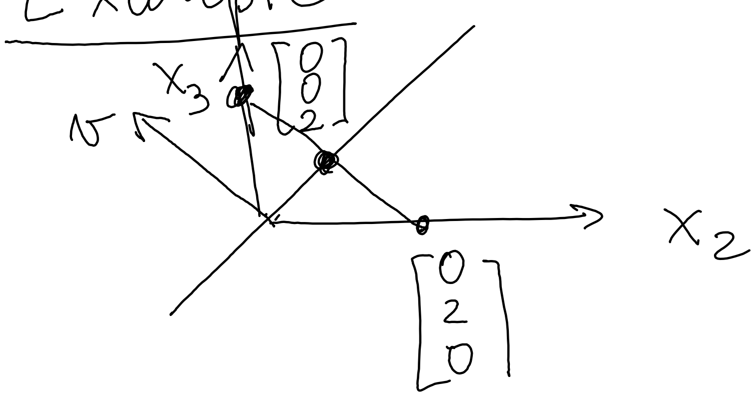
Tx



$$Tx = x - 2\left(\frac{v \cdot x}{v \cdot v}\right) v$$

Example

$v = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$



$\mathrm{proj}_P \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

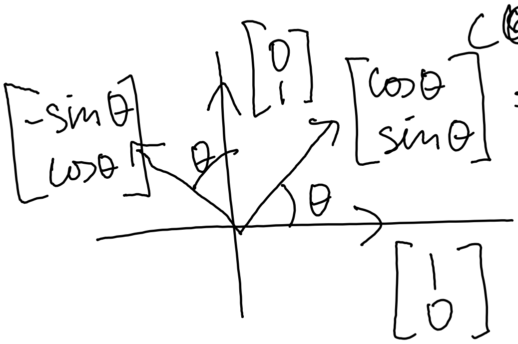
$\Big(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \cdot x \Big) \begin{bmatrix} 0 \end{bmatrix}$

0 1

$$\text{proj}_P x = x - \left(\frac{v \cdot x}{v \cdot v}\right) v = x - \left(\frac{\begin{bmatrix}0\\-1\\1\end{bmatrix}\cdot x}{\begin{bmatrix}0\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}0\\-1\\1\end{bmatrix}}\right)\begin{bmatrix}0\\-1\\1\end{bmatrix} =$$

$$= \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} - \frac{(-x_2+x_3)}{2}\begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}x_1\\ \frac{x_3+x_2}{2}\\ \frac{x_2+x_3}{2}\end{bmatrix}$$

<u>Rotations</u> R = rotating a vector in $\mathbb{R}^2$ counter clockwise by an angle :



$$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} = R\begin{bmatrix}1\\0\end{bmatrix}$$

$$R\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

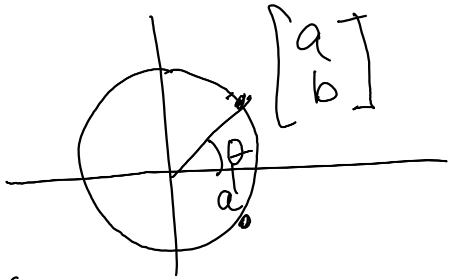
$$R = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \qquad Rx = \begin{bmatrix}(\cos\theta)x_1 - (\sin\theta)x_2\\ (\sin\theta)x_1 + (\cos\theta)x_2\end{bmatrix}$$

rotating $\frac{\pi}{6}$ $\quad R = \begin{bmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{bmatrix}$

<u>Obs</u>: R is a rotation if $R = \begin{bmatrix}a & -b\\ b & a\end{bmatrix}$

for some a & b such that $a^2+b^2=1$

for some ...



If $a, b \in \mathbb{R}$, not both 0, then

$$R = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \sqrt{a^2+b^2} \begin{bmatrix} \frac{a}{\sqrt{a^2+b^2}} & \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{bmatrix}$$

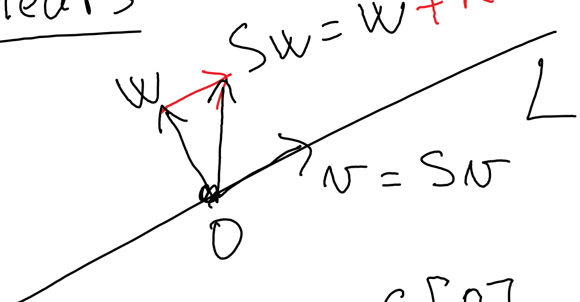
Let $c = \frac{a}{\sqrt{a^2+b^2}}$ & $d = \frac{b}{\sqrt{a^2+b^2}}$, then $c^2 + d^2 = 1$

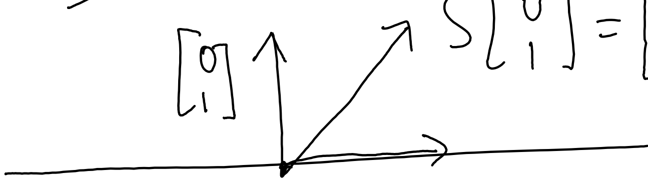
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \underset{\sqrt{a^2+b^2}}{\lambda} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}$$

rotation, then streching or compressing

Shears

S = shear





$$S\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + k\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = S\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix}1\\0\end{bmatrix} = S\begin{bmatrix}1\\0\end{bmatrix}$$

$$S = \begin{bmatrix}1 & k\\0 & 1\end{bmatrix} \qquad S\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1 + k x_2\\x_2\end{bmatrix}$$

Done with section 2.2

Composition of linear transformations

$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^p \qquad S: \mathbb{R}^p \longrightarrow \mathbb{R}^k$$

then $S \circ T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$

$$(S \circ T)(x) = S(T(x))$$

Claim: If both S & T are linear transformations, so is $S \circ T$

proof: $(S \circ T)(x+y) = S(T(x+y)) = S(T(x) + T(y)) = S(T(x)) + S(T(y)) = (S \circ T)(x) + (S \circ T)(y)$

$(S \circ T)(\lambda x) = S(T(\lambda x)) = S(\lambda T(x)) = \lambda S(T(x)) = \lambda (S \circ T)(x)$