<u>Example</u>

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 0 \\ 2x_1 + 2x_2 + x_3 + x_4 &= 1 \\ x_3 + 2x_4 &= -1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 1 & 0 \\ 2 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right] \; R_2 - 2R_1 \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & \boxed{3} & -1 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right]$$

$$R_2/3 \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & \boxed{1} & -1/3 & 1/3 \\ 0 & 0 & 1 & 2 & -1 \end{array}\right] \quad R_3 - R_2 \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1/3 & 1/3 \\ 0 & 0 & 0 & \boxed{7/3} & -4/3 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 1 & 0 \\ 0 & 0 & \boxed{1} & -1/3 & 1/3 \\ 0 & 0 & 0 & 1 & -4/7 \end{array}\right] \quad \begin{array}{l} R_1 - R_3 \\ R_2 + \frac{R_3}{3} \end{array} \left[\begin{array}{cccc|c} \boxed{1} & 1 & -1 & 0 & 4/7 \\ 0 & 0 & 1 & 0 & 1/7 \\ 0 & 0 & 0 & \boxed{1} & -4/7 \end{array}\right]$$

$$R_1 + R_2 \left[\begin{array}{cccc|c} \boxed{1} & 1 & 0 & 0 & 5/7 \\ 0 & 0 & \boxed{1} & 0 & 1/7 \\ 0 & 0 & 0 & \boxed{1} & -4/7 \end{array}\right]$$

$$\begin{aligned} x_1 &= 5/7 - t \\ x_2 &= t \\ x_3 &= 1/7 \\ x_4 &= -4/7 \end{aligned} \qquad X = \begin{bmatrix} 5/7 \\ 0 \\ 1/7 \\ -4/7 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

2) $x_1 + x_2 = 1$ $\quad \left[\begin{array}{cc|c} 1 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \end{array}\right]$

3) $x_1 + x_2 = 1$ $\quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & 2 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 3 \end{array}\right]$

$-x_1 - x_2 = 2$

there are no solutions $\qquad \left[\begin{array}{cc|c} \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} \end{array}\right]$

<u>Theorem</u>: A linear system is inconsistent if the
rrE form of the augmented matrix has a pivot
in the last column. If the last column is not a pivot
column, the system is consistent. In this case, if
all the variables are leading variables, Then there is
only one solution. Otherwise, there is an infinite
number of solutions.

<u>Def</u>: Let $A \in \mathbb{R}^{k \times n}$. rref(A) is the rre form of A.
, that is, the matrix we obtain after we apply Gaussian
elimination to transform A to a RRE matrix.

<u>Ex</u> $A = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$

<u>Def</u>: A matrix. rank of A = rank(A) =
number of leading 1's in rref(A).

number of leading 1's in rref(A).

Ex: $\operatorname{rank}\left(\begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}\right) = 2$

Obs: $A \in \mathbb{R}^{k \times n}$.

1) $\operatorname{rank}(A) \le n$ and $\operatorname{rank}(A) \le k$

2) $Ax = b$. $A \in \mathbb{R}^{k \times n}$. If $Ax = b$ is inconsistent, then rref[A | b] has a leading 1 in the last column, then # of leading 1's in rref(A) =

= # of leading 1's in rref([A | b]) $- 1 \le k - 1$

then $\operatorname{rank}(A) < k$

3) If $Ax = b$ has one solution then $\operatorname{rank} A = n$

$(A \in \mathbb{R}^{k \times n})$

4) If $Ax = b$ has an ∞ # of solutions, then

$\operatorname{rank}(A) < n$

5) If $\operatorname{rank}(A) = k \Rightarrow Ax = b$ is consistent

6) If $\operatorname{rank}(A) < n \Rightarrow Ax = b$ is inconsistent

or it has an ∞ # of solutions

7) If $\operatorname{rank}(A) = n \Rightarrow Ax = b$ is inconsistent or it has exacty one solution.

<u>Ex:</u> $x_1 = 1$

$x_2 = 1$

$x_1 + x_2 = 3$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

$$A \in \mathbb{R}^{\overset{k}{3} \times \overset{n}{2}}$$

<u>Th:</u> 1) If $Ax = b$ has exactly one solution, then then $\operatorname{rank}(A) = n$

$\operatorname{rank}(A) \le k$

$\underset{\#\text{ of unknowns}}{n} \le \underset{\#\text{ of equations}}{k}$

<u>Ex:</u> $x_1 + x_2 = 1$

$-x_1 + x_2 = 2$

$2x_2 = 3$

$x_2 = 3/2$

$x_1 = -1/2$

3 eq

2 unknowns

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 2 \end{bmatrix} \in \mathbb{R}^{3 \times 2}$$

<u>Th:</u> If $Ax = b$ has fewer equation than unknowns

then, if it is consistent, it has an ∞ # of solutions.

<u>Obs</u>: $Ax=b$ $A \in \mathbb{R}^{n \times n}$. this system has exacty one solution if and only if

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = I = \text{identity matrix}$$

<u>Def</u>: $x, y \in \mathbb{R}^n$. the dot product of x and y is

$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

<u>Obs</u>: $x, y \in \mathbb{R}^n$ $\quad x^T y = [x_1 \; x_2 \cdots x_n] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x \cdot y$

<u>Obs</u>: $A \in \mathbb{R}^{k \times n}$ $\quad A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_k^T \end{bmatrix}$ $\quad a_i^T$ is the i^{th} row of A. $x \in \mathbb{R}^n$ $\quad Ax = \begin{bmatrix} a_1^T \\ \vdots \\ a_k^T \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_k^T x \end{bmatrix} =$

$= \begin{bmatrix} a_1 \cdot x \\ a_2 \cdot x \end{bmatrix}$ <u>Ex</u>: $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$ $\quad a_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} a_1 \cdot x \\ a_2 \cdot x \\ \vdots \\ a_n \cdot x \end{bmatrix}$$ <u>Ex</u>: $A = \begin{bmatrix} \\ 1 & -1 & 1 \end{bmatrix}$ $a_1 = \begin{bmatrix} \\ 0 \end{bmatrix}$

$a_1^T = [2 \ 1 \ 0]$

<u>Obs</u>: Ix $I \in \mathbb{R}^{n \times n}$ I identity $\in \mathbb{R}^n$

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = I \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$I = \begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} = \begin{bmatrix} e_1^T \\ e_2^T \\ \vdots \\ e_n^T \end{bmatrix}$$

<u>Ex</u>:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$I = \begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} e_1^T \\ e_2^T \end{bmatrix}$$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th} \qquad e_i^T x = e_i \cdot x = x_i \qquad x \in \mathbb{R}^n$$

$$Ix = \begin{bmatrix} e_1^T x \\ \vdots \\ e_n^T x \end{bmatrix} = \begin{bmatrix} e_1 \cdot x \\ \vdots \\ e_n \cdot x \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x$$

$\in \mathbb{R}^k$ $a_1, a_2, \ldots, a_n \in \mathbb{R}^k$.

Def: $b \in \mathbb{R}^k$. $a_1, a_2, \dots, a_n \in \mathbb{R}^k$.
We say that b is a linear combination of the vectors $a_1, a_2, \dots, a_n$ if there exists $x_1, x_2, \dots, x_n$ such that

$$b = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

Ex: $\begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ thus, $\begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Obs: b is a linear combinatoon of $a_1, \dots, a_n$
$\iff \exists\, x_1, \dots, x_n \in \mathbb{R}$ such that $b = x_1 a_1 + \dots + x_n a_n$
$= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$\iff b = Ax$ is constistent, where $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$

Obs: $A \in \mathbb{R}^{k \times n}$. $x, y \in \mathbb{R}^n$. $\lambda \in \mathbb{R}$. Then

1) $A(x+y) = Ax + Ay$

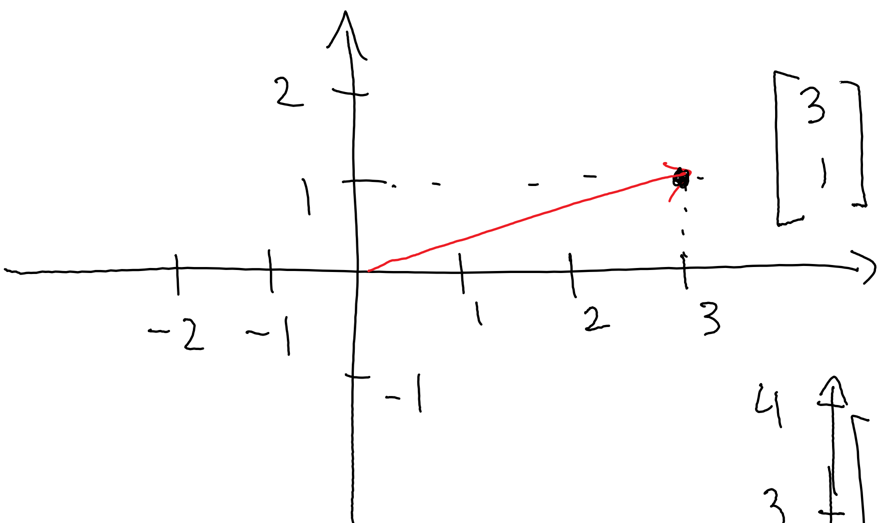
2) $A(\lambda x) = \lambda(Ax)$

Geometric interpretation to addition of vectors and scalar-vector multiplication.

Parallelogram rule

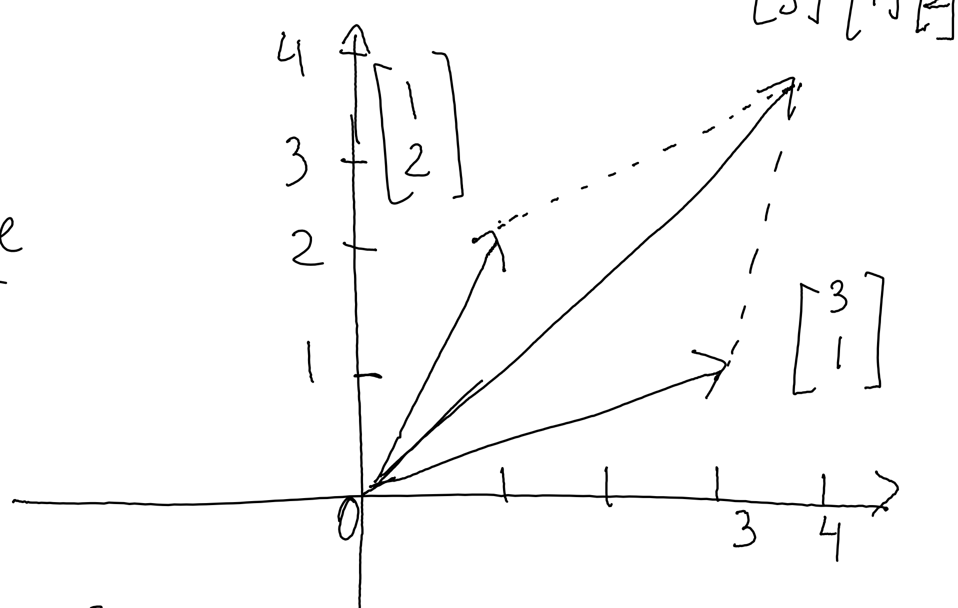
We can plot vectors. 2-vectors are identified with points (2-vectors = vectors with two component Ex $\begin{bmatrix}-5\\1\end{bmatrix}$) in the plane. Ex

We draw vectors as points or as arrows.

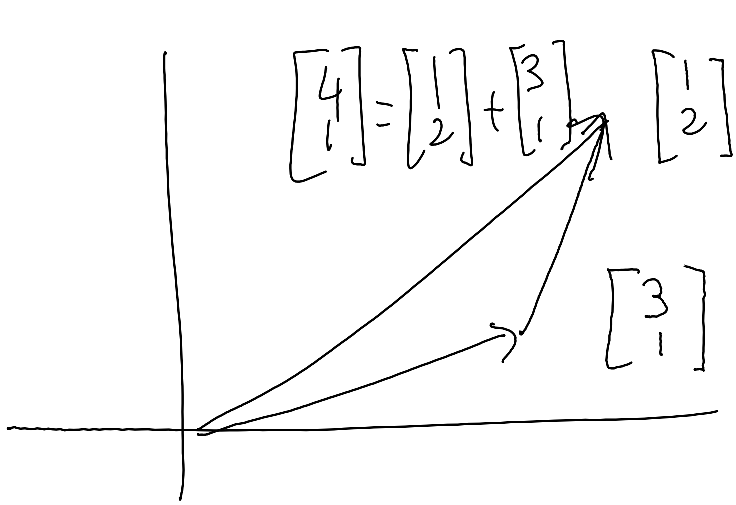


Parallelogram rule

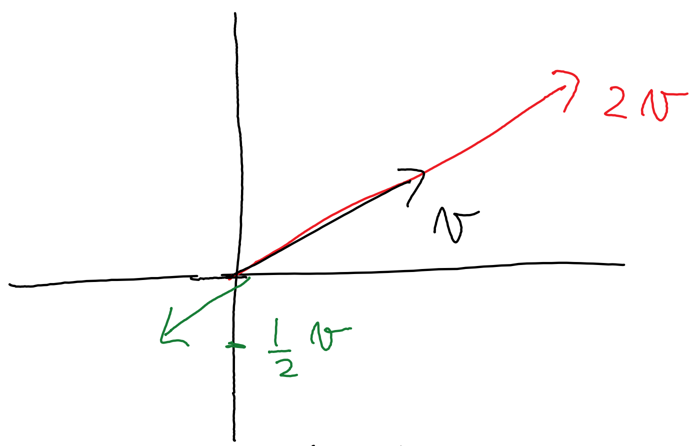
$\begin{bmatrix}4\\3\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$



$\begin{bmatrix}4 \end{bmatrix} - \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}3\end{bmatrix} \quad \begin{bmatrix}1\end{bmatrix}$



Scalar-vector multiplication



End of Ch 1. Start Ch 2

Def: $T: \mathbb{R}^n \longrightarrow \mathbb{R}^k$. We say that T is a

($\mathbb{R}^n$: domain; $\mathbb{R}^k$: codomain or range.)

linear transformation if there exists $A \in \mathbb{R}^{k \times n}$ such that $T(x) = Ax$ for all $x \in \mathbb{R}^n$.

$T: \mathbb{R}^3 \to \mathbb{R}^2$

<u>Example</u> $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix}$ $T: \mathbb{R}^3 \to \mathbb{R}^2$

$$T(x) = Ax = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 - x_3 \\ 3x_2 + x_3 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 - x_3 \\ 3x_2 + x_3 \end{bmatrix}$$ for example:

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2(1) + 2 - 0 \\ 3(2) + 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

<u>Notation</u> If $T(x) = Ax$, we say that A is the matrix associated with T, or A is the matrix of T.

<u>Ex</u>: $T: \mathbb{R}^n \to \mathbb{R}^n$ $T(x) = x$

note that $$Ix = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

then $T(x) = x$ $T(x) = Ix$, thus, the identity is a linear transformation.

$[0]$

Identity is a linear transformation.

Recall $e_i \in \mathbb{R}^n$ $\quad e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{th}$

Obs: $A \in \mathbb{R}^{k \times n}$ $\quad e_i \in \mathbb{R}^n$

$$A e_i = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0a_1 + 0a_2 + \cdots + 1a_i + 0a_{i+1} + \cdots + 0a_n = a_i$$

$A e_i = i^{th}$ column of A

Ex. $\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 0 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{e_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Obs: If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ is a linear transformation, then $T(x) = Ax$ for some A. Then the i^{th} column of A is $T(e_i)$, i.e.

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$$

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$$

<u>Obs</u>: $T: \mathbb{R}^n \to \mathbb{R}^k$ such that, for all $x, y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, the following is satisfied

1) $T(x+y) = T(x) + T(y)$
2) $T(\lambda x) = \lambda(T(x))$

then T is a linear transformation, and it's matrix is

$$\begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$$

<u>proof</u>. $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{e_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{e_2} + \cdots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

$$\boxed{T(x)} = T(x_1 e_1) + T(x_2 e_2) + \cdots + T(x_n e_n) =$$

$$= x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) =$$

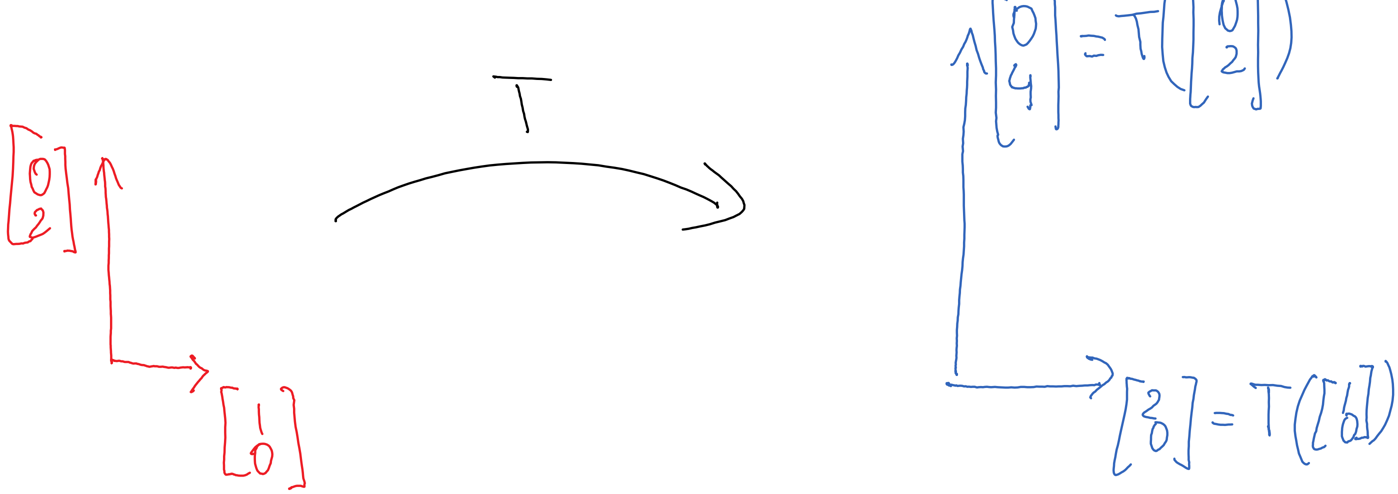
$$= \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \begin{bmatrix} \vdots \\ x_n \end{bmatrix}$$

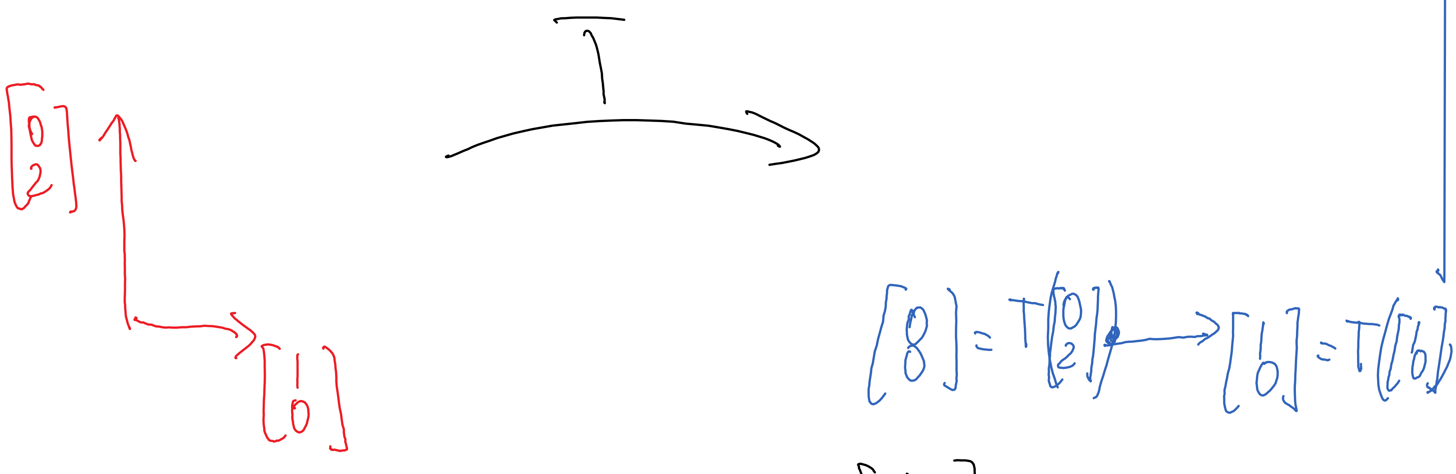
<u>Geometry</u>: Linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$

$T(x) = Ax \quad A \in \mathbb{R}^{2\times 2}$

<u>Ex</u>: 1) $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad T(x) = Ax = 2x$

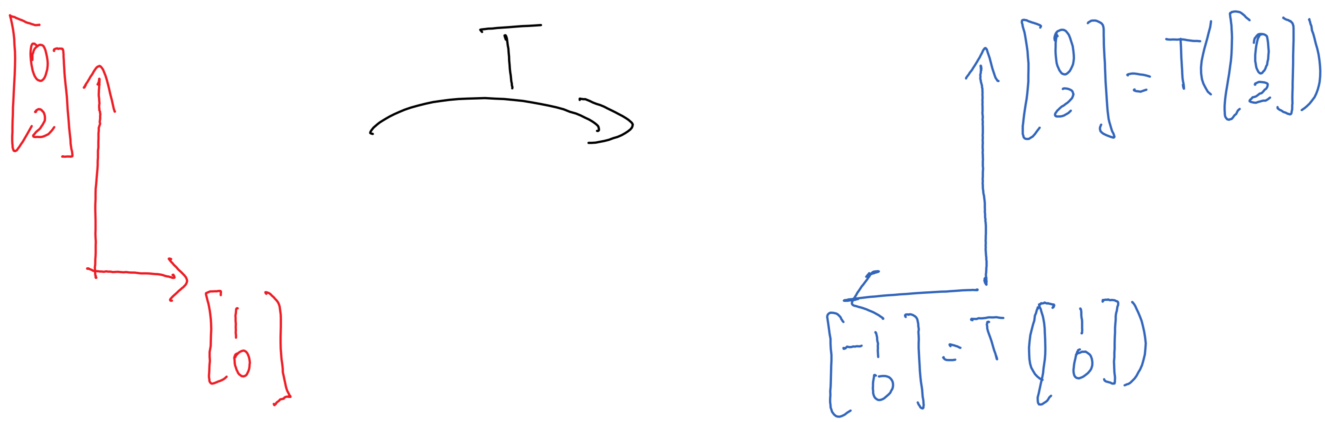


2) $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad T(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$



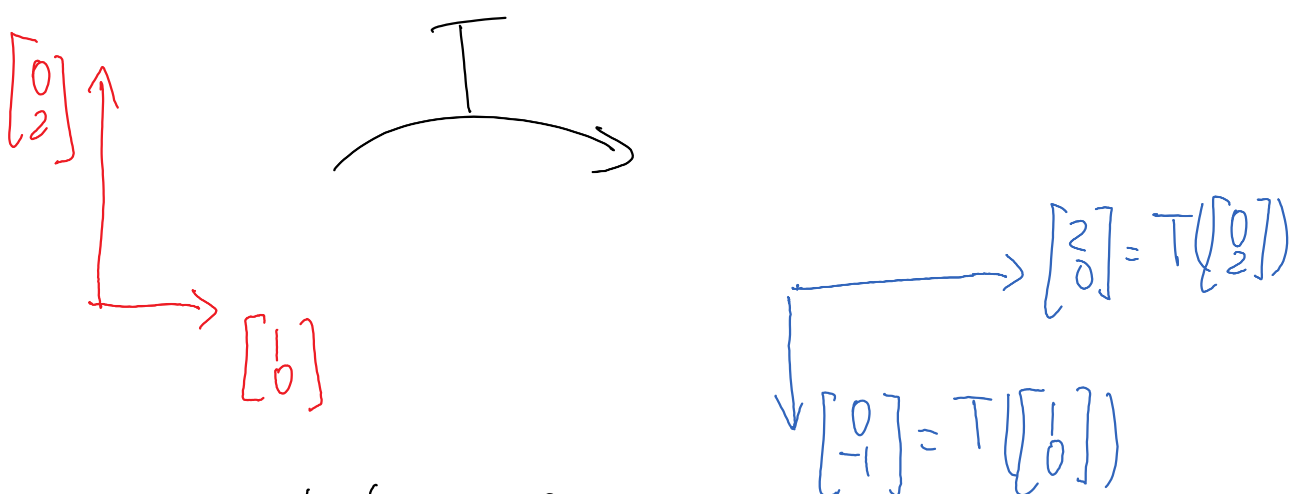
3) $C = \begin{bmatrix} -1 & 0 \\ \cdot & \cdot \end{bmatrix} \qquad T(x) = \begin{bmatrix} -x_1 \\ x \end{bmatrix}$

3) $C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $T(x) = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$



Reflection about the vertical axis.

4) $D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$



Clockwise rotation of 90° or $\frac{\pi}{2}$

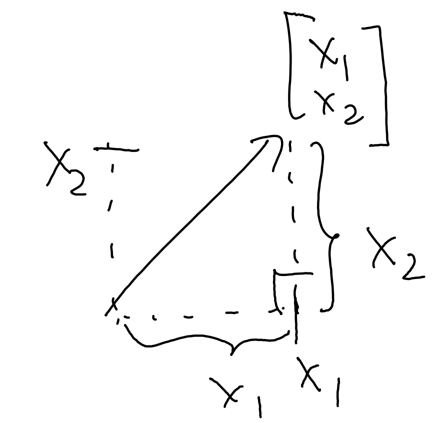
<u>Reminder</u> 1) $x, y \in \mathbb{R}^n$

$$x \cdot y = x^t y = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

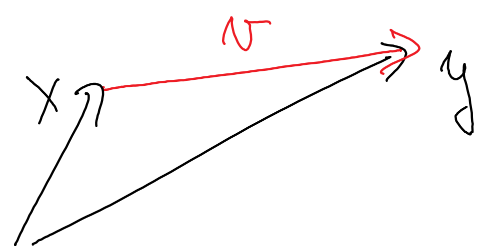
$x \cdot y = x^T y = [x_1 \cdots x_n] \begin{bmatrix} \vdots \\ y_n \end{bmatrix}$

2) $x \in \mathbb{R}^n$. The norm or length of x is

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$



3)



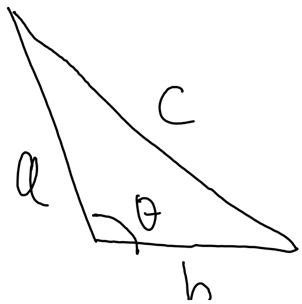
$$x + v = y$$
$$v = y - x$$

distance between x and y = length of $y - x$ =

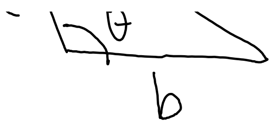
$= \text{norm of } y - x = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}$

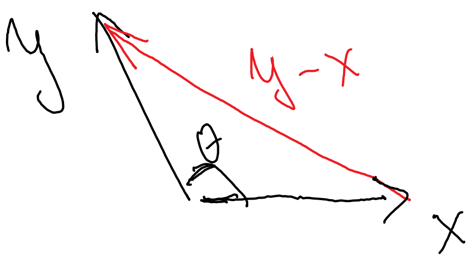
4) $\|x\| = \sqrt{x \cdot x}$ for all $x \in \mathbb{R}^n$

5)



$$c^2 = a^2 + b^2 - 2ab\cos\theta$$





$$\|y-x\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\|\cos\theta$$

$$(y-x)\cdot(y-x) = x\cdot x + y\cdot y - 2\|x\|\|y\|\cos\theta$$

$$\cancel{y\cdot y} - 2x\cdot y + \cancel{x\cdot x} = \cancel{x\cdot x} + \cancel{y\cdot y} - 2\|x\|\|y\|\cos\theta$$

$$\boxed{\cos\theta = \frac{x\cdot y}{\|x\|\,\|y\|}}$$

<u>Obs</u>: $x, y, z \in \mathbb{R}^n$ $\lambda \in \mathbb{R}$

1) $(x+y)\cdot z = x\cdot z + y\cdot z$

2) $(\lambda x)\cdot y = \lambda(x\cdot y)$

3) $x\cdot y = y\cdot x$

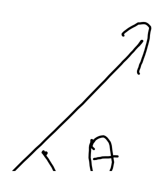
<u>Def</u>: 1) $x, y \in \mathbb{R}^n$. The angle between them is the only θ such that $0 \le \theta \le \pi$ and

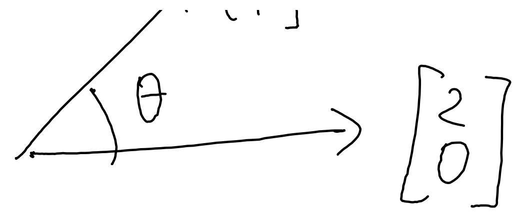
$$\cos\theta = \frac{x\cdot y}{\|x\|\,\|y\|} \quad (\text{if } x \neq 0 \ \& \ y \neq 0)$$

2) $x, y \in \mathbb{R}^n$. We say that x & y are orthogonal (perpendicular) if $x\cdot y = 0$.

<u>Example</u>

$\begin{bmatrix}1\\1\end{bmatrix}$





$$\cos\theta = \frac{\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}2\\0\end{bmatrix}}{\left\|\begin{bmatrix}1\\1\end{bmatrix}\right\|\left\|\begin{bmatrix}2\\0\end{bmatrix}\right\|} = \frac{2}{\sqrt{2}\ 2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} = \cos\left(\frac{\pi}{4}\right)$$

$$\theta = \frac{\pi}{4}$$