Eigenvalues & eigenvectors

Def: Let A be an n by n matrix. Let $p(\lambda) = \det(A-\lambda I)$. Note that $p(\lambda)$ is a polynomial of degree n. Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be the different eigenvalues of A. Note that λ_i may a complex number.

Then

$$p(\lambda) = (-1)^n (\lambda-\lambda_1)^{n_1} (\lambda-\lambda_2)^{n_2} \cdot \dots \cdot (\lambda-\lambda_r)^{n_r}$$

$n_1, \dots, n_r$ are positive integers.

$\boxed{n_i \text{ is called the algebraic multiplicity of } \lambda_i.}$

Recall $E_{\lambda_i} = \ker(A-\lambda_i I) = \{$ all eigenvectors of eigenvalue λ_i and the vector $0\}$

$\boxed{\text{The geometric multiplicity of } \lambda_i \text{ is } \dim(E_{\lambda_i})}$

Obs:

1) $1 \leq$ geometric multiplicity of $\lambda_i \leq$ algebraic mutipli-

1) $1 \le$ geometric multiplicity of $\lambda_i \le$ algebraic mutiplicity of λ_i

2) $n_1 + n_2 + \cdots + n_r =$ sum of all the algebraic multiplicities $= n$.

Examples: 1) Find the algebraic & geometric multiplicities of the eigenvalues of $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

$$p(\lambda) = \det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & -\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda)(-\lambda)(1-\lambda) = -(\lambda-1)^2 \lambda$$

Eigenvalues	algebraic mult	eigenvectors	geo mult
0	1	$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$	1
1	2	$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	1

$\lambda = 0$

$$A - 0I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -t$

$x_2 = t$

$x_3 = 0$

$$v = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 1 \quad A - I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v = t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Recall: $A \in \mathbb{R}^{n \times n}$. A is diagonalizable if $\exists$ n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$

If this is the case, let $S = [v_1 \ v_2 \ \ldots \ v_n]$

Let $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$ $\quad \boxed{Av_i = \lambda_i v_i}$

$\boxed{AS = SD}$. Which is the same as say $\exists$ S non-singular & D diagonal such that

$$\boxed{S^{-1} A S = D}$$

Obs: A is diagonalizable $\Longleftrightarrow$ the sum of the geometric multiplicities equals n $\Longleftrightarrow$ all the geometric multiplicities equal the algebraic multiplicities.

Example: $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$p(\lambda) = \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{bmatrix} = (1-\lambda) \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} +$$

$$\det \begin{bmatrix} 1 & 1-\lambda \\ 1 & 0 \end{bmatrix} = (1-\lambda)^3 + (-1)(1-\lambda) = (1-\lambda)\left[(\lambda-1)^2 - 1\right]$$

$$= (1-\lambda)\,(\lambda^2 - 2\lambda) = (-1)\,\lambda\,(\lambda-2)(\lambda-1)$$

Example $A = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix}$

$$p(\lambda) = \lambda^2 + 4 = 0 \implies \lambda = \pm 2i$$

$\lambda = 2i \quad A - 2iI = \begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{-4}{-2i} \cdot \frac{i}{i} \\ 1 & -2i \end{bmatrix}$

$$\begin{bmatrix} 1 & -2i \\ 1 & -2i \end{bmatrix} \quad \begin{bmatrix} 1 & -2i \\ 0 & 0 \end{bmatrix} \quad v = t\begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

Eigenvalue	Eigenvector	Alg mult	Geo mult
$2i$	$\begin{bmatrix} 2i \\ 1 \end{bmatrix}$	1	1
$-2i$	$\begin{bmatrix} -2i \\ 1 \end{bmatrix}$	1	1

Symmetric matrices

$A^T = A$ symmetric Ex: $\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$

Obs 1) The eigenvalues of symmetric matrices are real.

proof: $A v = \lambda v \quad v \neq 0$

$A \bar{v} = \bar{\lambda} \bar{v}$

$\bar{v}^T A v = \bar{v}^T \lambda v$

$v^T A \bar{v} = v^T \bar{\lambda} \bar{v}$

$$\Rightarrow v^T A v = v \quad \lambda v$$

$$\bar{v}^T A v = (\bar{v}^T A v)^T = v^T A^T (\bar{v}^T)^T = v^T A \bar{v}$$

$$\bar{v}^T A v = (\bar{v}^T v)\lambda \qquad = \qquad (v^T \bar{v})\bar{\lambda} = v^T A \bar{v}$$

$$\lambda \overbrace{(\bar{v}^T v)} = \bar{\lambda} \overbrace{(v^T \bar{v})}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \bar{v} = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \vdots \\ \bar{v}_n \end{bmatrix} \qquad \bar{v}^T v = \underbrace{|v_1|^2 + |v_2|^2 + \cdots + |v_n|^2}_{>0 \quad v \neq 0}$$

$$\bar{v}^T v = v^T \bar{v}$$

then $\lambda = \bar{\lambda}$ $\qquad a+bi = a-bi \Rightarrow b=0$

Obs 2) A symmetric $\quad Av_1 = \lambda_1 v_1 \quad Av_2 = \lambda_2 v_2$

$v_1 \neq 0 \quad v_2 \neq 0 \quad \lambda_1 \neq \lambda_2$. then $v_1 \cdot v_2 = 0$

proof:

$$v_1^T A v_2 = v_1^T A^T v_2 = (v_2^T A v_1)^T = v_2^T A v_1 = v_2^T \lambda_1 v_1 = (v_2^T v_1)\lambda_1$$

$$v_1^T A v_2 = v_1^T \lambda_2 v_2 = (v_1^T v_2)\lambda_2$$

$(v_1 \cdot v_2)\lambda_1 = (v_1 \cdot v_2)\lambda_2$ since $\lambda_1 \neq \lambda_2$ then $v_1 \cdot v_2 = 0$

<u>Obs</u>: If A is symmetric, then A is diagonalizable.

<u>Def</u>: We say that A is orthogonally diagonalizable if $\exists\ v_1, \ldots, v_n$ orthonormal that are eigenvectors of A.

<u>Obs</u>: A is orthogonally diagonalizable $\iff$ $\exists\ S$ orthogonal and D diagonal such that $S^T A S = D$

Recall $S \in \mathbb{R}^{n \times n}$ S is orthogonal means the columns of S form an orthonormal set.

Recall S orthogonal $S \in \mathbb{R}^{n \times n}$ then $S^{-1} = S^T$

Theorem: $A \in \mathbb{R}^{n\times n}$ symmetric $\Longleftrightarrow$ A is orthogonally diagonalizable.

Recall: 1) A symmetric means $A^T = A$

2) A orthogonally diagonalizable means there exists S orthogonal and D diagonal such that

$$S^T A S = D$$

In this case, the columns of S are eigenvectors of A and the diagonal elements of D are the eigenvalues of A.

Obs: Given A symmetric, orthogonally diagonalize A means to find S orthogonal and D diagonal such that $S^T A S = D$.

Ex: Orthogonally diagonalize $\begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix}$

$$p(\lambda) = \det \begin{bmatrix} 4-\lambda & 2 \\ 2 & 7-\lambda \end{bmatrix} = (4-\lambda)(7-\lambda) - 4 = 0$$

$$\lambda^2 - 11\lambda + 24 = 0$$

$$\frac{11 \pm \sqrt{121-96}}{2} = \frac{11 \pm 5}{2} = \begin{cases} 8 \\ 3 \end{cases}$$

$$\lambda_1 = 3 \quad A - \lambda_1 I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \frac{v_1}{\|v_1\|} = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

$$\lambda_2 = 8 \quad A - \lambda_2 I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \frac{v_2}{\|v_2\|} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$S = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$$

$$S^T A S = D$$

$$\begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} =$$

$$\begin{bmatrix} -\frac{6}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\ \dots & \dots \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \dots & \dots \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ \dots & \dots \end{bmatrix} = D \checkmark$$

$$\begin{bmatrix} \frac{\ }{\sqrt{5}} & \frac{\ }{\sqrt{5}} \\ \frac{8}{\sqrt{5}} & \frac{16}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{\ }{\sqrt{5}} & \frac{\ }{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 8 \end{bmatrix} = D \checkmark$$

<u>Example</u> Orthogonally diagonilaze $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$\det \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} = (1-\lambda) \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} -$$

$$- \det \begin{bmatrix} 1 & 1 \\ 1 & 1-\lambda \end{bmatrix} + \det \begin{bmatrix} 1 & 1-\lambda \\ 1 & 1 \end{bmatrix} = (1-\lambda)\left((1-\lambda)^2 - 1\right)$$

$$- (1-\lambda-1) + (1+\lambda-1) = (1-\lambda)(\lambda^2 - 2\lambda) + 2\lambda = \lambda\left[(1-\lambda)(\lambda-2) + 2\right]$$

$$= \lambda\left[\lambda - 2 - \lambda^2 + 2\lambda + 2\right] = \lambda(\lambda^2 + 3\lambda) = \lambda^2(-\lambda+3) = 0$$

$\lambda_1 = 0$ & $\lambda_2 = 3$

$\lambda = 0$

$$A - 0I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad v = t_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvectors of eigenvalue $\lambda = 0$ $\quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Apply Gram-Schmidt

$$\frac{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\|} = \boxed{\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}}$$

$$u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$w_2 = \frac{u_2}{\|u_2\|} = \frac{\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}}{\sqrt{\frac{3}{2}}} = \boxed{\begin{bmatrix} -\sqrt{\frac{1}{6}} \\ -\sqrt{\frac{1}{6}} \\ \sqrt{\frac{2}{3}} \end{bmatrix}}$$

$\lambda_2 = 3$

$$A - 3I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & -\frac{3}{2} \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \frac{v}{\|v\|} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\sqrt{\frac{1}{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\sqrt{\frac{1}{6}} & \frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$S^T A S = D$$

Singular values

Def: $A \in \mathbb{R}^{k \times n}$ matrix. The singular values of A are

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$$

Where $\sigma_i = \sqrt{\lambda_i}$ and λ_i is an eigenvalue of $A^T A$

λ_i is reapeated n_i times where n_i is the multiplicity of λ_i

Example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix}$ Find the singular values of A

$$A^T A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}$$

$$(5-\lambda)(1-\lambda)-1 = \lambda^2 - 6\lambda + 4 = 0$$

$$\frac{6 \pm \sqrt{36-16}}{2} = \frac{6 \pm \sqrt{20}}{2} = 3 \pm \sqrt{5}$$

singular values $\sqrt{3+\sqrt{5}}$ & $\sqrt{3-\sqrt{5}}$

<u>Obs</u>: If $v \neq 0$ $A^T A v = \lambda v \Rightarrow \lambda \geq 0$

Multiply by v^T $\quad v^T A^T A v = \lambda v^T v$

$$(Av)^T Av = \lambda v^T v$$

$$\underbrace{\|Av\|^2}_{\geq 0} = \lambda \underbrace{\|v\|^2}_{>0} \Rightarrow \lambda \geq 0$$

<u>Obs</u>: $\sigma_1^2 \geq \dots \geq \sigma_n^2 \geq 0$ eigenvalues of A^TA.

$v_1, \dots, v_n$ orthonormal & eigenvectors of A^TA

$A^TAv_i = \sigma_i^2 v_i$ $\quad$ multiply at the left by v_j^T

$$v_j^T A^T A v_i = \sigma_i^2 \underbrace{v_j^T v_i}_{=0} \qquad i \neq j$$

$$(Av_j)^T (Av_i) = 0$$

Thus, $Av_1, Av_2, \dots, Av_n$ is orthogonal

$$\|Av_i\|^2 = v_i^T A^T A v_i = \sigma_i^2 \|v_i\|^2$$

$$\| \qquad\qquad A^T A v_i = \sigma_i^2 v_i$$

$$(Av_i)^T (Av_i)$$

$$\boxed{\|A v_i\| = \sigma_i \|v_i\| = \sigma_i}$$

<u>def</u>: $u_i = \dfrac{A v_i}{\sigma_i}$ if $\sigma_i \neq 0$

$$\sigma_1 \geq \dots \geq \sigma_r > 0 \qquad \sigma_{r+1} = \dots = \sigma_n = 0$$

$u_1, \dots, u_r$ are orthonormal

$$A \begin{bmatrix} v_1 \dots v_r \, v_{r+1} \dots v_n \end{bmatrix} = \begin{bmatrix} Av_1 & Av_2 & \dots & Av_r & Av_{r+1} & \dots & Av_n \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_1 u_1 & & \sigma_r u_r & 0 & \dots & 0 \end{bmatrix}$$

Let $u_{r+1}, \dots, u_k$ orthonormal such that

$u_1, \dots, u_k$ is orthonormal

$$A \begin{bmatrix} v_1 \dots v_n \end{bmatrix} = \begin{bmatrix} \sigma_1 u_1 \dots \sigma_r u_r \, 0 \dots 0 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} u_1 \cdots u_r & 0 \cdots 0 \end{bmatrix}}_{k} \Big\}k \quad \underbrace{\begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & 0 & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & 0 & & & 0 & \\ & & & & & \ddots \ 0 \end{bmatrix}}_{n} \Big\}k$$

$$= \begin{bmatrix} u_1 \cdots u_r \, u_{r+1} \cdots u_k \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \ \ddots \end{bmatrix}$$

$V = \begin{bmatrix} v_1 \cdots v_n \end{bmatrix}$ orthonormal basis of $\mathbb{R}^n$

$V \in \mathbb{R}^{n \times n}$ $\quad U = \begin{bmatrix} u_1 \cdots u_k \end{bmatrix}$ orthogonal $\mathbb{R}^{k \times k}$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix} \in \mathbb{R}^{k \times n}$$

$$\underset{k \times n}{A} \ \underset{n \times n}{V} = \underset{k \times k}{U} \ \underset{k \times n}{\Sigma}$$

This is called the singular value decomposi-

$k \times n$ $k \times n$ tion of A.