<u>Obs</u>: $x^2+1=0$ $\quad x^2=-1$ $\quad x=\pm i$

<u>Roots of degree two polynomials</u>

$ax^2+bx+c=0 \quad a,b,c\in\mathbb{R}$

$\Delta = b^2-4ac$

If $\Delta>0$ $\quad x=\dfrac{-b\pm\sqrt{\Delta}}{2a}$ $\quad$ 2 real roots

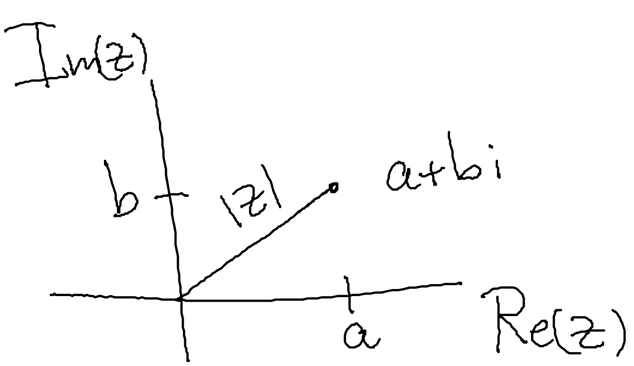
If $\Delta=0$ $\quad x=-\dfrac{b}{2a}$ $\quad$ 1 real root

If $\Delta<0$ $\quad x=\dfrac{-b\pm i\sqrt{-\Delta}}{2a}$ $\quad$ 2 complex roots.

<u>Def</u>: $z=a+bi$ $\quad a \,\&\, b\in\mathbb{R}$. The complex conjugate of z is $\bar{z}=a-bi$

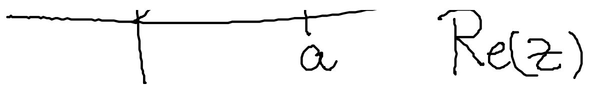
<u>Example</u>: $z=2+i$ $\quad \bar{z}=2-i$

<u>Obs</u>: $z\bar{z}=(a+bi)(a-bi)=a^2+b^2$ $\quad a,b\in\mathbb{R}$



$|z|=\sqrt{a^2+b^2}=\sqrt{z\bar{z}}$

<u>Def</u>: the modulus or absolute <u>value</u> of z is



absolute value of z is

$$|z| = \sqrt{z\bar{z}}$$

Def: $\mathbb{C}$ = set of complex numbers.

$\mathbb{C}^n$ = set of all n-vectors whose components are complex numbers.

$z \in \mathbb{C}^n$ means $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$ $\quad z_i \in \mathbb{C}$.

Obs: $z = a+bi \qquad w = c+di \qquad a, b, c, d \in \mathbb{R}$.

$$\frac{z}{w} = \frac{z\,\bar{w}}{w\,\bar{w}} = \frac{z\,\bar{w}}{|w|^2} = \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$$

Example: $\dfrac{1+2i}{3-4i} = \dfrac{(1+2i)(3+4i)}{(3-4i)(3+4i)} = \dfrac{(3-8)}{25} + i\,\dfrac{(4+6)}{25} =$

$$= -\frac{1}{5} + i\,\frac{2}{5}$$

Obs: $A \in \mathbb{R}^{n\times n}$. $v \in \mathbb{C}^n$ $\lambda \in \mathbb{C}$. $v \neq 0$.

v is an eigenvector with eigenvalue λ if

$$Av = \lambda v$$

Obs: Let w_1 & $w_2 \in \mathbb{C}$. then

$$\overline{w_1 + w_2} = \overline{w_1} + \overline{w_2}$$

$$\overline{w_1 w_2} = \overline{w_1}\ \overline{w_2}$$

Obs: Let $P(x)$ be a real polynomial. Let $z \in \mathbb{C}$

If $P(z) = 0$ then

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0 = 0$$

$$\boxed{0} = \overline{0} = \overline{P(z)} = \overline{a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0} =$$

$$= \overline{a_n}\ \overline{z}^n + \overline{a_{n-1}}\ \overline{z}^{n-1} + \dots + \overline{a_1}\ \overline{z} + \overline{a_0} =$$

$$= a_n \overline{z}^n + a_{n-1} \overline{z}^{n-1} + \dots + a_1 \overline{z} + a_0 = \boxed{P(\overline{z})}$$

Obs If $P(x)$ is a real polynomial & $z \in \mathbb{C}$ then z is a root of $P(x) \iff \overline{z}$ is a root of $P(x)$.

Example: $x^2 + 2x + 2 = 0$

$$\frac{-2 \pm \sqrt{4 - 4(2)}}{2} = \frac{-2 \pm i2}{2} = -1 \pm i$$

Obs: $A \in \mathbb{R}^{n \times n}$, $v \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$

$\boxed{Av = \lambda v}$ take complex conjugate

$\boxed{Av = \lambda v}$ take complex conjugate

$\overline{A}\,\overline{v} = \overline{\lambda}\,\overline{v}$ $\quad\lambda$ eigenvalue with eigenvector v

$\boxed{A\overline{v} = \overline{\lambda}\,\overline{v}}$ $\quad$ of the matrix $A \iff \overline{\lambda}$ eigen-value with eigenvector $\overline{v}$ of the matrix A.

Example: Find the eigenvalues & eigenvectors of

$$A = \begin{bmatrix} 6 & -1 \\ 5 & 4 \end{bmatrix}$$

$$P(\lambda) = \det\begin{bmatrix} 6-\lambda & -1 \\ 5 & 4-\lambda \end{bmatrix} = (6-\lambda)(4-\lambda) + 5 = \lambda^2 - 10\lambda + 29 =$$

$$= (\lambda-5)^2 + 4 = 0 \qquad \lambda - 5 = \pm 2i$$

$$\lambda = 5 \pm 2i$$

$\lambda = 5+2i$

$$A - \lambda I = \begin{bmatrix} 1-2i & -1 \\ 5 & -1-2i \end{bmatrix} \qquad (1-2i)v_1 - v_2 = 0$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} +1 \\ 1-2i \end{bmatrix}$$

$$a v_1 + b v_2 = 0 \qquad v = \begin{bmatrix} -b \\ a \end{bmatrix} \qquad \begin{aligned} v_1 &= -b \\ v_2 &= a \end{aligned}$$

Eigenvalues	Eigenvectors
$5+2i$	$\begin{bmatrix} 1 \\ 1-2i \end{bmatrix}$
$5-2i$	$\begin{bmatrix} 1 \\ 1+2i \end{bmatrix}$

Obs: $\lambda = 0$ is an eigenvalue of $A \iff A$ is singular

Obs: Assume A is not singular.

$$A v = \lambda v \quad v \neq 0$$
$$A^{-1}(Av) = A^{-1}(\lambda v)$$
$$v = \lambda A^{-1} v$$
$$\frac{1}{\lambda} v = A^{-1} v$$

A non-singular.
λ is an eigenvalue of A with eigenvector $v \iff \frac{1}{\lambda}$ is an eigenvalue of A^{-1} with eigenvector v.

Def: $A \in \mathbb{R}^{n \times n}$. We say that A is diagonalizable $\iff \exists\, v_1, \ldots, v_n$ linearly independent eigenvector of A. (in $\mathbb{R}$ or $\mathbb{C}$)

Obs: A diagonalizable $\iff$ $\exists\, v_1, \ldots, v_n$ l.i. and $\lambda_1, \ldots, \lambda_n$ such that $A v_i = \lambda_i v_i$ $\iff$

$\exists\, [v_1 \ldots v_n]$ non-singular such and $\lambda_1, \ldots, \lambda_n$ such that

$$A\begin{bmatrix} v_1 \ldots v_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 \ldots \lambda_n v_n \end{bmatrix} = \underbrace{\begin{bmatrix} v_1 \ldots v_n \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D}$$

$\iff$ $\exists$ S non-singular & D diagonal such that

$AS = SD$ $\iff$ A is similar to a diagonal matrix (this means $S^{-1}AS = D$ for some S non-singular & D diagonal.

Example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad (1-\lambda)^2 = 0$

$\lambda = 1$ ✓ $\quad A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable

$A = \lceil 1 \;\; 1 \rceil$ $\quad P = (1-\lambda)(2-\lambda) = 0$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \qquad P = (1-\lambda)(2-\lambda) = 0$$

$$\lambda = 1 \quad A - I = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\lambda = 2 \quad A - 2I = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$S^{-1} A S = D \qquad \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = D \checkmark$$

<u>Obs</u>: $A v_i = \lambda_i v_i \quad 1 \le i \le r \qquad \lambda_i \ne \lambda_j \quad i \ne j$

$v_i \neq 0$. then $\{v_1, \dots, v_r\}$ is linearly independent

<u>proof</u>: Assume v_j is a linear combination of $v_1, \dots, v_{j-1}$ but $v_1, \dots, v_{j-1}$ is linearly independent.

$(*) \quad v_j = a_1 v_1 + \dots + a_{j-1} v_{j-1}$ multiply by A

$$Av_j = a_1 Av_1 + \cdots + a_{j-1} Av_{j-1}$$

(1) $\lambda_j v_j = a_1 \lambda_1 v_1 + \cdots + a_{j-1} \lambda_{j-1} v_{j-1}$

multiply $\circledast$ by λ_j

(2) $\lambda_j v_j = a_1 \lambda_j v_1 + \cdots + a_{j-1} \lambda_j v_{j-1}$

(1)-(2) $\quad 0 = a_1(\lambda_1 - \lambda_j) v_1 + \cdots + a_{j-1}(\lambda_{j-1} - \lambda_j) v_{j-1}$

Since $v_1, \ldots, v_{j-1}$ is linearly independent,

$$a_1 \underbrace{(\lambda_1 - \lambda_j)}_{\neq 0} = \cdots = a_{j-1} \underbrace{(\lambda_{j-1} - \lambda_j)}_{\neq 0} = 0$$

Then $a_1 = \cdots = a_{j-1} = 0 \Rightarrow v_j = 0$ contradiction

Then $v_1, \ldots, v_n$ are linearly independent.