Eigenvalues & Eigenvectors

$A \in \mathbb{R}^{n\times n}$. λ is an eigenvalue of A if there exist $v \in \mathbb{R}^n$ such that $v \neq 0$ and $Av = \lambda v$. In this case we say v is an eigenvector of A of eigenvalue λ.

Obs: λ is an eigenvalue of A if and only if $\det(A-\lambda I)=0$.

Def: Let $A \in \mathbb{R}^{n\times n}$. $P(\lambda) = \det(A-\lambda I)$

$P(\lambda)$ is called the characteristic polynomial of A.

Obs: $P(\lambda)$ is a polynomial of degree n.

Finding eigenvalues & eigenvectors

Step 1: Solve for λ $P(\lambda)=0$. These are the eigenvalues.

Step 2: For each λ found in step 1, solve for v

$(A-\lambda I)v=0$. These are the eigenvectors, except $v=0$

Example: $A=\begin{bmatrix}1 & 2 & 1\\ 6 & -1 & 0\\ -1 & -2 & -1\end{bmatrix}$. Find the eigenvalues & eigenvector of A.

$$0=P(\lambda)=\det(A-\lambda I)=\det\begin{bmatrix}1-\lambda & 2 & 1\\ 6 & -1-\lambda & 0\\ -1 & -2 & -1-\lambda\end{bmatrix}=$$

$$(1-\lambda)\det\begin{bmatrix}-1-\lambda & 0\\ -2 & -1-\lambda\end{bmatrix}-2\det\begin{bmatrix}6 & 0\\ -1 & -1-\lambda\end{bmatrix}+$$

$$\det\begin{bmatrix}6 & -1-\lambda\\ -1 & -2\end{bmatrix}=(1-\lambda)(-1-\lambda)^2-2(6)(-1-\lambda)+(-12-(-1)(-1-\lambda))$$

$$=(1-\lambda)(1+2\lambda+\lambda^2)+12+12\lambda-12-(\lambda+1)=$$

$$=1+2\lambda+\lambda^2-\lambda-2\lambda^2-\lambda^3+12\lambda-\lambda-1=$$

$$=-\lambda^3-\lambda^2+12\lambda=\lambda(-\lambda^2-\lambda+12)=-\lambda(\lambda+4)(\lambda-3)$$

Eigenvalues: $-\lambda(\lambda+4)(\lambda-3)=0$

$$\boxed{\lambda=-4,\ 0,\ 3}$$

$\boxed{\lambda = -4, 0, 3}$

Eigenvectors: $\lambda = -4$ $\quad (A - \lambda I)v = 0 \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

$$A - (-4)I = \begin{bmatrix} 5 & 2 & 1 \\ 6 & 3 & 0 \\ -1 & -2 & 3 \end{bmatrix}$$

$v_1 = -t$

$v_2 = 2t$

$v_3 = t$

$$\begin{bmatrix} 1 & 2/5 & 1/5 \\ 0 & 3/5 & -6/5 \\ 0 & -8/5 & 16/5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \qquad v = t \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

$t \in \mathbb{R}$

Def: If λ is an eigenvalue of A, then

$E_\lambda = \ker(A - \lambda I)$ = eigenspace of eigenvalue λ of A

When we are asked for the eigenvectors of eigenvalue λ we give a basis of E_λ.

Back to example $\quad A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$

Eigenvalue	-4	0	3
Eigenvector	$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -1 \\ -6 \\ 13 \end{bmatrix}$	$\begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

$\lambda = 0 \quad A - 0I = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -13 & -6 \end{bmatrix}$

$\lambda = 6 \quad A - 6I \quad \begin{bmatrix} 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -13 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 1/13 \\ 0 & 1 & 6/13 \\ 0 & 0 & 0 \end{bmatrix} \qquad v = t \begin{bmatrix} -1/13 \\ -6/13 \\ 1 \end{bmatrix}$$

$$\lambda = 3 \qquad A - 3I = \begin{bmatrix} -2 & 2 & 1 \\ 6 & -4 & 0 \\ -1 & -2 & -4 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & -1/2 \\ 0 & 2 & 3 \\ 0 & -3 & -9/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix} \qquad v = t \begin{bmatrix} -1 \\ -3/2 \\ 1 \end{bmatrix}$$

<u>Obs</u>: 1) The characteristic polynomial $P(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n if $A \in \mathbb{R}^{n \times n}$.

2) An n by n matrix has at most n eigenvalues.

<u>Example</u>: $A = \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix}$ Find the eigenvalues

and eigenvectors.

$$P(\lambda) = \det A - \lambda I = \det \begin{bmatrix} 3-\lambda & 4 \\ -1 & 7-\lambda \end{bmatrix} = (3-\lambda)(7-\lambda)+4$$

$$= \lambda^2 - 10\lambda + 25 = (\lambda - 5)^2 = 0$$

Eigenvalues	5
Eigenvectors	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$A - 5I = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \qquad v = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

<u>Example</u>: $A = \begin{bmatrix} 9 & 1 & 1 \\ 1 & 9 & 1 \\ 1 & 1 & 9 \end{bmatrix}$

$$P(\lambda) = \begin{bmatrix} 9-\lambda & 1 & 1 \\ 1 & 9-\lambda & 1 \\ 1 & 1 & 9-\lambda \end{bmatrix} = (9-\lambda)\left[(9-\lambda)^2 - 1\right] - 1(9-\lambda-1) +$$

$$+ 1 - (9-\lambda) = (9-\lambda)\ (9-\lambda+1)(9-\lambda-1) + (\lambda-8) - (8-\lambda)$$

$$= (9-\lambda)(\lambda-10)(\lambda-8) + (\lambda-8) + (\lambda-8) =$$

$$= (\lambda-8)\left\{-\lambda^2 + 19\lambda - 90 + 2\right\}$$

$$= (\lambda-8)(-1)\left(\lambda^2 - 19\lambda + 88\right) = (-1)(\lambda-8)^2(\lambda-11)$$

$$(\lambda-11)(\lambda-8)$$

Eigenvalue	8	11
Eigenvector	$\begin{bmatrix}-1\\1\\0\end{bmatrix}$ $\begin{bmatrix}-1\\0\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\lambda = 8$ $\quad A - 8I = \begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$ $\quad v = t_1\begin{bmatrix}-1\\1\\0\end{bmatrix} + t_2\begin{bmatrix}-1\\0\\1\end{bmatrix}$

$v_1 = -t_1 - t_2$

$v_2 = t_1$

$v_3 = t_2$

$\lambda = 11$ $\quad A - 11I = \begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}$

$\begin{bmatrix}1&-1/2&-1/2\\0&-3/2&3/2\\0&3/2&-3/2\end{bmatrix}$ $\begin{bmatrix}1&0&-1\\0&1&-1\\0&0&0\end{bmatrix}$ $\quad v = t\begin{bmatrix}1\\1\\1\end{bmatrix}$

Complex numbers

i is a new number

$i^2 = -1$

Def: A complex number is a number of the form $a+bi$, where a and b are real numbers.

Example: $2+3i$

Operations with complex numbers

Addition: $(a+bi)+(c+di) = (a+c) + (b+d)i$

Example: $(-2+3i)+(4-2i) = 2+i$

Multiplication: $(a+bi)(c+di) = ac + adi + bic + bidi = (ac-bd) + (ad+bc)i$

Example $(2-i)(3+2i) = 6+4i-3i+2 = 8+i$

Def: 1) $\mathbb{C}$ denotes the set of all complex numbers. $z \in \mathbb{C}$, then $z = a+bi$ with $a, b \in \mathbb{R}$

a is called the real part of z $\quad a = \mathrm{Re}(z)$

b is called the imaginary part of z $\quad b = \mathrm{Im}(z)$

$$z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$$

Ex: $\mathrm{Re}(3+2i) = 3 \qquad \mathrm{Im}(3+2i) = 2$