Dot products in vector spaces other than $\mathbb{R}^n$.

$\langle f,g\rangle \quad f,g \in V$.

<u>Def</u>: Norm. Let $f \in V$, V vector space with a dot product $\langle \, , \, \rangle$. The norm of f is

$$\|f\| = \sqrt{\langle f,f\rangle}$$

<u>Ex</u>: $V = C[0,1] = \{f:[0,1] \to \mathbb{R} : f \text{ is continuous}\}$..

$$\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx$$

$$\|\sin(\pi x)\| = \sqrt{\int_0^1 \sin^2(\pi x)\,dx} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

<u>Def</u>: $f,g \in V$. V vector space with inner product $\langle \, , \, \rangle$. We say that f & g are orthogonal if

$$\langle f,g\rangle = 0.$$

<u>Ex</u>: $V = C[0,1]$

$$\langle \sin(\pi x), \cos(\pi x) \rangle = \int_0^1 \sin(\pi x)\cos(\pi x)\,dx = 0$$

<u>Def</u>: V an inner product space.

$g_1, \dots, g_r$ a basis of S subspace of V.

It is an orthonormal basis if $\langle g_i, g_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

In this case,

$$\text{proj}_S(f) = \langle g_1, f\rangle g_1 + \langle g_2, f\rangle g_2 + \dots + \langle g_r, f\rangle g_r$$

Recall $\text{proj}_S(f)$ is the vector in S closest to f.

<u>Example</u>: $f = e^x \qquad V = C[0,1]$

$S = \text{Span}\{1, x\}$ Compute $\text{proj}_S(e^x)$.

Apply Gram-Schmidt to $\{1, x\}$

$$u_1 = \frac{1}{\|1\|} = \frac{1}{\int_0^1 (1)^2\,dx} = \frac{1}{1} = 1$$

$$w_2 = x - \text{proj}_{\text{span}\{1\}} x = x - \langle x, 1\rangle 1 = x - \left(\int_0^1 x\,dx\right) 1 =$$

$$= x - \frac{1}{2}$$

$$u_2 = \frac{x - \frac{1}{2}}{} = \frac{x - \frac{1}{2}}{} = \frac{x - \frac{1}{2}}{} = 2\sqrt{3}\left(x - \frac{1}{2}\right)$$

$$u_2 = \frac{x-\frac{1}{2}}{\|x-\frac{1}{2}\|} = \frac{x-\frac{1}{2}}{\sqrt{\int_0^1 (x-\frac{1}{2})^2 dx}} = \frac{x-\frac{1}{2}}{\sqrt{\frac{1}{12}}} = 2\sqrt{3}\,(x-\tfrac{1}{2})$$

$$\text{proj}_S e^x = \langle 1, e^x\rangle 1 + \langle 2\sqrt{3}(x-\tfrac{1}{2}), e^x\rangle 2\sqrt{3}(x-\tfrac{1}{2}) =$$

$$= (e-1) + 12(x-\tfrac{1}{2}) \int_0^1 (x-\tfrac{1}{2}) e^x dx$$

$$(x-\tfrac{1}{2})e^x \Big|_0^1 - \int_0^1 e^x dx = \tfrac{1}{2}(e+1) - (e-1) = \frac{3-e}{2}$$

$$\boxed{\text{proj}_S(e^x) = (e-1) + 6(3-e)(x-\tfrac{1}{2})}$$

<u>Fourier Analysis</u>

$V = C[-\pi, \pi]$ $\qquad \langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$

$T_n = \text{Span}\{1, \sin(x), \cos(x), \ldots\ldots, \sin(nx), \cos(nx)\}$

$$\|\sin(nx)\| = \|\cos(nx)\| = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(x)\, dx} = 1$$

$$\|1\| = \sqrt{\frac{1}{\pi}\int_{-\pi}^{\pi} 1^2 dx} = \sqrt{2}$$

Orthonormal basis of T_n is

$\frac{1}{\sqrt{2}}$, $\sin(x)$, $\cos(x)$, ..., $\sin(nx)$, $\cos(nx)$

Let $f \in C[-\pi, \pi]$

$$f_n = \text{proj}_{T_n}(f) = a_0 \frac{1}{\sqrt{2}} + b_1 \sin(x) + \dots + b_n \sin(nx) + c_1 \cos(x) + \dots + c_n \cos(nx)$$

$$a_0 = \left\langle \frac{1}{\sqrt{2}}, f \right\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(x)}{\sqrt{2}}\, dx$$

$$b_j = \langle \sin(jx), f(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(jx)\, dx$$

$$c_j = \langle \cos(jx), f(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(jx)\, dx$$

<u>Obs</u>:

$g_1, \dots, g_n$ orthonormal

$$\|(g_1 \cdot f) g_1 + \dots + (g_n \cdot f) g_n\|^2 = (g_1 \cdot f)^2 + \dots + (g_n \cdot f)^2$$

$$\|\text{proj}_{T_n} f\|^2 = a_0^2 + b_1^2 + c_1^2 + \dots + b_n^2 + c_n^2$$

Fact $\|f - \text{proj}_{T_n} f\|^2 \xrightarrow[n \to \infty]{} 0$

~~Ex~~ thus $\|f\|^2 = \lim_{n\to\infty} \|\text{proj}_{V_{T_n}} f\|^2$

Ex $f(x) = x$

$\langle 1, x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0$

$\langle \sin(nx), x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{1}{\pi}\left\{ -\frac{x\cos(nx)}{n}\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos(nx)\,dx \right\}$

$= \frac{1}{\pi n}\left(-\pi(-1)^n - \pi(-1)^n\right) = \frac{2}{n}(-1)^{n+1}$

$\langle \cos(nx), x\rangle = 0$

$\|x\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{3\pi}\pi^3 = \frac{2}{3}\pi^2$

$\frac{2}{3}\pi^2 = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \cdots\right)$

Start of chapter 6

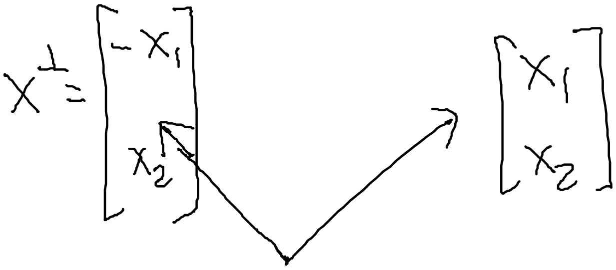
Determinants:

Def $A \in \mathbb{R}^{2\times 2}$ $\det(A) = a_{11}a_{22} - a_{21}a_{12}$

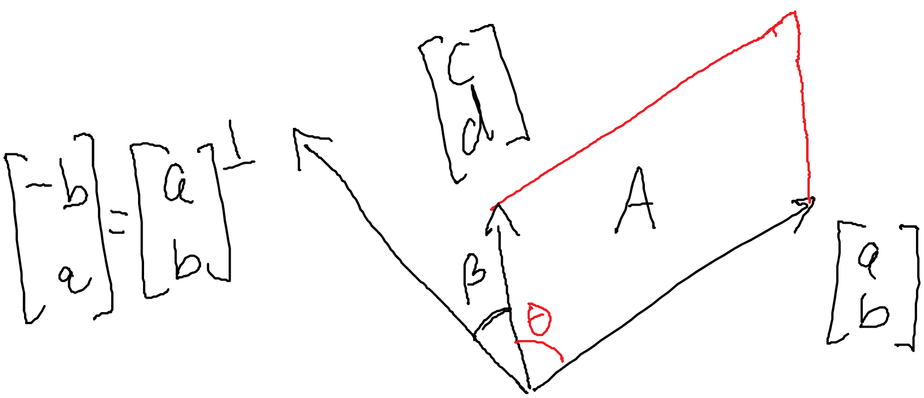
$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$

$x = \begin{bmatrix} x_1 \\ \end{bmatrix}$ $x^\perp = \begin{bmatrix} -x_2 \\ \end{bmatrix}$ $x \cdot x^\perp = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad x^{\perp} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \qquad x \cdot x^{\perp} = 0$$



$$\det \begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc = \begin{bmatrix} -b \\ a \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}^{\perp} \cdot \begin{bmatrix} c \\ d \end{bmatrix}$$



$$\text{Area } A = \sin\theta \left\| \begin{bmatrix} c \\ d \end{bmatrix} \right\| \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \begin{bmatrix} a \\ b \end{bmatrix}^{\perp} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \det \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix}^{\perp} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\| \left\| \begin{bmatrix} c \\ d \end{bmatrix} \right\| \cos(\beta)$$

Area $= |\det(A)|$ Area of parallelogram with

... of A being 2 sides of the

the columns of A being 2 sides of the parallelogram.

Def: $A \in \mathbb{R}^{n\times n}$ matrix

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} + \cdots - (-1)^n a_{1n}\det A_{1n}$$

A_{ij} = Matrix A minus i^{th} row & j^{th} column

Example

$$\det\begin{bmatrix}1 & -1 & 2\\ 0 & -2 & -1\\ 3 & 1 & 2\end{bmatrix} = 1\det\begin{bmatrix}-2 & -1\\ 1 & 2\end{bmatrix} -$$

$$(-1)\det\begin{bmatrix}0 & -1\\ 3 & 2\end{bmatrix} + 2\det\begin{bmatrix}0 & -2\\ 3 & 1\end{bmatrix} = (-2)2 - 1(-1) +$$

$$+ 0(2) - 3(-1) + 2\left(0(1) - 3(-2)\right) = -4 + 1 + 3 + 12 = \boxed{12}$$

Example: V = polynomials of degree ~~to~~ or less and or is the pol 0

$\int^1 p(x)\,dx = 0.$

$\int_0^1 p(x)\,dx = 0.$

1) V is a linear space $\quad V \subset \mathbb{P}_2$ we show V is a subspace of $\mathbb{P}_2$

a) $0 \in V$ ✓

b) $f \,\&\, g \in V \Rightarrow \int_0^1 f(x)\,dx = \int_0^1 g(x)\,dx = 0$

$\Rightarrow \int_0^1 (f+g)(x)\,dx = \int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx = 0+0=0$

$\Rightarrow f+g \in V$ ✓

c) $f \in V \,\&\, \lambda \in \mathbb{R} \Rightarrow \int_0^1 (\lambda f)(x)\,dx =$

$= \lambda \int_0^1 f(x)\,dx = \lambda 0 = 0 \Rightarrow \lambda f \in V$ ✓

then V is a subspace.

Basis of V. $\quad p = a + bx + cx^2$

$$\boxed{0 = \int_0^1 p(x)\,dx = a + \frac{b}{2} + \frac{c}{3}}$$

V is a subspace of $P_2 = \{ p \text{ poly} : \deg p \le 2 \text{ or } p = 0 \}$

$a = -t_1/2 - t_2/3$

$b = t_1$

$c = t_2$

$c = t_2$

$p = a+bx+cx^2 = \left(-\frac{t_1}{2}-\frac{t_2}{3}\right) + t_1 x + t_2 x^2$

$= t_1 \underbrace{\left(-\frac{1}{2}+x\right)} + t_2 \underbrace{\left(-\frac{1}{3}+x^2\right)}$

Basis $= \left\{ \left(-\frac{1}{2}+x\right), \left(-\frac{1}{3}+x^2\right) \right\}$

$\dim V = 2.$

<u>Example</u> $T: \mathbb{R}^2 \longrightarrow \mathbb{P}_2$

$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = (a+b)x^2 + bx + a$

$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = T\left(a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = aT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + bT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

then $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ span $\mathrm{Im}(T)$

remove some vector as necessary.

$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = x^2+1 \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = x^2+x$

[illegible] $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

<u>Determinants</u>: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

in $\mathbb{R}^3$. Let $x, y \in \mathbb{R}^3$. The cross product of x and y is

$$x \times y = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} =$$

$$= e_1 (x_2 y_3 - y_2 x_3) - e_2 (x_1 y_3 - y_1 x_3) + e_3 (x_1 y_2 - y_1 x_2)$$

<u>Example</u>

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \det \begin{bmatrix} e_1 & e_2 & e_3 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix} =$$

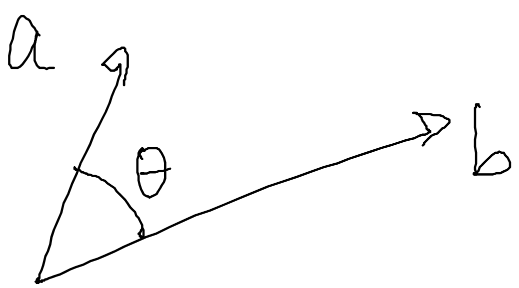
$$e_1 \det \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - e_2 \det \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} + e_3 \det \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$$

$$= e\,1 - e_2(-2) + e_3(-1) = \begin{bmatrix}1\\2\\-1\end{bmatrix}$$

Obs: 1) $a, b \in \mathbb{R}^3 \Rightarrow (a \times b) \perp a$

& $(a \times b) \perp b$. $\qquad$ $\Vert$ $\quad (a \times b) \cdot a = 0$

2) $\|a \times b\| = \|a\| \, \|b\| \sin\theta$



Obs: $A = \begin{bmatrix} a^T \\ b^T \\ c^T \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_2 \\ c_1 & c_2 & c_3 \end{bmatrix}$

$$\det \begin{bmatrix} a^T \\ b^T \\ c^T \end{bmatrix} = a \cdot b \times c$$

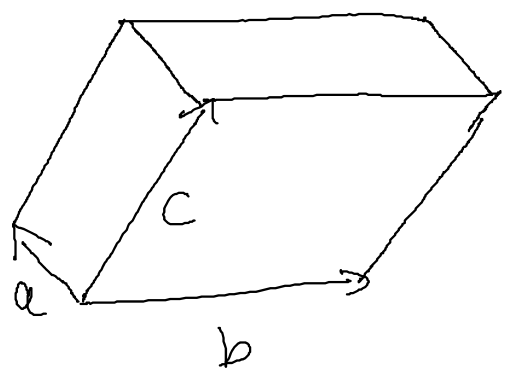
Example: $a = \begin{bmatrix}1\\0\\1\end{bmatrix} \quad b = \begin{bmatrix}-1\\1\\0\end{bmatrix} \quad c = \begin{bmatrix}2\\0\\1\end{bmatrix}$

$$\det\begin{bmatrix} a^T \\ b^T \\ c^T \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = -1$$

$$b \times c = \det\begin{bmatrix} e_1 & e_2 & e_3 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = e_1 + e_2 - 2e_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$$a \cdot (b \times c) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = -1$$

<u>Obs</u>: $A = \begin{bmatrix} a & b & c \end{bmatrix} \in \mathbb{R}^{3 \times 3}$



Volume of parallelepiped $= \left| \det\begin{bmatrix} a & b & c \end{bmatrix} \right|$

<u>Properties</u>: $A, B \in \mathbb{R}^{n \times n}$

1) $\det A = \det A^T$
2) $\det(AB) = \det A \det B$
3) If A is triangular, then $\det A = a_{11} a_{22} \cdots a_{nn}$

4) A – matrix from A after removing i^{th} row

4) A_{ij} = matrix from A after removing i^{th} row & j^{th} column

$$\det(A) = (-1)^{i+1} a_{i1} \det(A_{i1}) + (-1)^{i+2} a_{i2} \det(A_{i2}) + \dots$$

$$\dots + (-1)^{i+n} a_{in} \det(A_{in}) = (-1)^{1+j} a_{1j} \det(A_{1j}) +$$

$$+ (-1)^{2+j} a_{2j} \det(A_{2j}) + \dots + (-1)^{n+j} a_{nj} \det(A_{nj})$$

<u>Ex</u>: $j=3$

$$\det \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & -1 & 0 & 2 \\ 1 & -3 & 1 & 0 \\ 5 & 0 & 0 & -1 \end{bmatrix} = \det \begin{bmatrix} 1 & -2 & 1 \\ 0 & -1 & 2 \\ 5 & 0 & -1 \end{bmatrix} =$$

$$= \det \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} + 5 \det \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix} = 1 + 5(-3) = -14$$

<u>Eigenvalues</u> & <u>Eigenvectors</u>

<u>Def</u>: $A \in \mathbb{R}^{n \times n}$. We say that $v \in \mathbb{R}^n$ is an eigenvector of A with eigenvalue λ if $v \neq 0$ and $Av = \lambda v$.

Example: $\begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \end{bmatrix}$

Example: $\underbrace{\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underbrace{2}_{\lambda} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{v}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$ with eigenvalue 2.

Finding eigenvalues and eigenvectors

$Av = \lambda v$ same as

$Av = \lambda I v$ same as

$Av - \lambda I v = 0$ Same as

$(A - \lambda I) v = 0$

λ is an eigenvalue $\iff \exists\, v \neq 0$ $v \in \mathbb{R}^n$ such that $(A-\lambda I)v = 0 \iff \lambda$ such that $\ker(A - \lambda I) \neq \{0\}$

$\iff \lambda$ such that $A - \lambda I$ does not have an inverse

Obs: $A \in \mathbb{R}^{n \times n}$. Then A is invertible $\iff$

$\det(A) \neq 0$.

Th: $A \in \mathbb{R}^{n\times n}$. λ is an eigenvalue of A

$\iff \det(A-\lambda I)=0$

Def: Let $A \in \mathbb{R}^{n\times n}$. $P(\lambda)=\det(A-\lambda I)$ is called the characteristic polynomial of A.