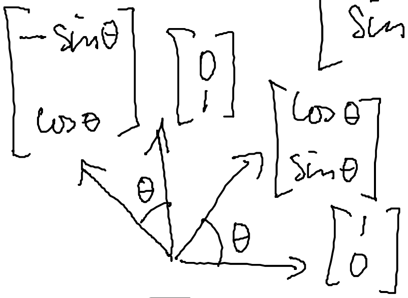
Def: 1) $T: \mathbb{R}^n \to \mathbb{R}^n$ linear transformation.
We say that T is orthogonal if $\|T(x)\| = \|x\|$ for all $x \in \mathbb{R}^n$.

2) $A \in \mathbb{R}^{n\times n}$. We say that A is orthogonal if $T(x) = Ax$ is an orthogonal linear transformation.

Ex: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\quad T(x) = Ax$



T & A are orthogonal.

$$\boxed{\|T(x)\|^2} = \|Ax\|^2 = \left\| \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2$$

$$\left\| \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix} \right\|^2 = (x_1\cos\theta - x_2\sin\theta)^2 +$$

$$+ (x_1\sin\theta + x_2\cos\theta)^2 = x_1^2\cos^2\theta - 2x_1x_2\cos\theta\sin\theta + x_2^2\sin^2\theta +$$

$$+ x_1^2\sin^2\theta + 2x_1x_2\sin\theta\cos\theta + x_2^2\cos^2\theta = x_1^2 + x_2^2 = \boxed{\|x\|^2}$$

Th: $T: \mathbb{R}^n \to \mathbb{R}^n$ orthogonal. If $x \cdot y = 0$

then $T(x) \cdot T(y) = 0$

<u>th</u>: $T: \mathbb{R}^n \to \mathbb{R}^n$. T is orthogonal $\iff$

$T(e_1), \ldots, T(e_n)$ is an orthonormal basis.

<u>proof</u>: $\Rightarrow$) $\|T(e_j)\| \underset{\substack{\uparrow \\ \text{because } T \\ \text{is orthogonal}}}{=} \|e_j\| = 1$ ✓

If $i \neq j \Rightarrow e_i \cdot e_j = 0 \underset{\substack{\uparrow \\ \text{because } T \\ \text{is orthogonal}}}{\Longrightarrow} T(e_i) \cdot T(e_j) = 0$ ✓

$\Leftarrow$) Let $x \in \mathbb{R}^n$. Note $x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$

$\|T(x)\|^2 = \| x_1 T(e_1) + x_2 T(e_2) + \cdots + x_n T(e_n) \|^2 =$

$$= x_1^2 \underbrace{(T(e_1) \cdot T(e_1))}_{=1} + x_2^2 \underbrace{(T(e_2) \cdot T(e_2))}_{=1} + \cdots + x_n^2 \underbrace{(T(e_n) \cdot T(e_n))}_{=1}$$

$T(e_1), \ldots, T(e_n)$ is orthogonal

then $\|T(x)\| = \|x\|$

<u>Ex</u>: $A = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ the columns of

A is an orthonormal basis of $\mathbb{R}^4$. $T(x) = Ax$

the columns of A are $T(e_1), T(e_2), T(e_3), T(e_4)$. then

T is an orthogonal linear transformation.

<u>Th</u> a) $A, B \in \mathbb{R}^{n\times n}$. A, B both orthogonal.
the AB is also orthogonal

<u>proof</u>: Let $x \in \mathbb{R}^n$ $\|(AB)x\| = \|A(Bx)\| \underset{\uparrow}{=} \|Bx\| \underset{\uparrow}{=} \|x\|$

because B is orthogonal.

because A is orthogonal

b) $A \in \mathbb{R}^{n\times n}$ orthogonal $\Longrightarrow A^{-1}$ is also orthogonal

<u>proof</u>: $T(x) = Ax$ $T(e_1), \dots, T(e_n)$ are linearly independent because they are a basis of $\mathbb{R}^n$, they are also the columns of A. then A has an inverse.

Let $x \in \mathbb{R}^n$. $\boxed{\|A^{-1}x\| \underset{\uparrow}{=} \|A(A^{-1}x)\| = \|x\|}$.

because A is orthogonal

Thus, A^{-1} is orthogonal.

<u>Def</u>: $A \in \mathbb{R}^{n\times m}$.

1) $A^T \in \mathbb{R}^{m\times n}$ $[A^T]_{ij} = [A]_{ji}$

Ex $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$

2) A is symmetric if $A = A^T$

2) A is symmetric if $A = A^T$

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 7 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & -2 \\ -2 & 7 \end{bmatrix}$$

3) A is skew-symmetric if

$$A^T = -A \qquad A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

Th: $A \in \mathbb{R}^{n \times n}$. A orthogonal $\Longleftrightarrow$ $A^T = A^{-1}$

$A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ orthogonal $\Longleftrightarrow$

$a_1, a_2, \dots, a_n$ is an orthonormal basis of $\mathbb{R}^n$

$$\Longleftrightarrow \qquad a_i \cdot a_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$$A^T A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} =$$

$$\begin{bmatrix} a_1^T a_1 & a_1^T a_2 & \dots & a_1^T a_n \\ a_2^T a_1 & a_2^T a_2 & \dots & a_2^T a_n \\ \vdots & \vdots & & \vdots \\ n & & & \end{bmatrix} = \begin{bmatrix} a_1 \cdot a_1 & a_1 \cdot a_2 & \dots & a_1 \cdot a_n \\ a_2 \cdot a_1 & a_2 \cdot a_2 & \dots & a_2 \cdot a_n \\ & & & \\ a_n \cdot a_1 & a_n \cdot a_2 & \dots & a_n \cdot a_n \end{bmatrix}$$

$= A^{-1}$

$$A^T A = I \iff a_i \cdot a_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

$\iff$ A is orthogonal

<u>Th</u>: Properties of transpose:

1) $(A+B)^T = A^T + B^T$
2) $(\lambda A)^T = \lambda A^T$
3) $(AB)^T = B^T A^T \qquad A \in \mathbb{R}^{l \times n} \quad B \in \mathbb{R}^{n \times s}$

$$\left[(AB)^T\right]_{ij} = \left[AB\right]_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}$$

$$\left[B^T A^T\right]_{ij} = \sum_{k=1}^{n} \left[B^T\right]_{ik} \left[A^T\right]_{kj} = \sum_{k=1}^{n} b_{ki} a_{jk} \quad \checkmark$$

4) $\text{rank}(A^T) = \text{rank}(A)$
5) $(A^T)^{-1} = (A^{-1})^T$

$$A^T (A^{-1})^T = (A^{-1} A)^T = I^T = I \quad \checkmark$$

then $(A^{-1})^T = (A^T)^{-1}$

<u>Obs</u>: $u \in \mathbb{R}^n \quad \|u\| = 1 \quad L = \text{span}\{u\}$

$$\text{proj}_L(x) = (u \cdot x)\, u = u u_1 x_1 + u u_2 x_2 + \cdots + u u_n x_n$$

$$= \begin{bmatrix} u_1 u & u_2 u & \cdots & u_n u \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \end{bmatrix} =$$

$$= \begin{bmatrix} u_1 u & u_2 u & \cdots & u_n u \end{bmatrix} \begin{bmatrix} \vdots \\ x_n \end{bmatrix} =$$

$$= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} [u_1 \cdots u_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} =$$

$$= u\, u^T x \qquad \mathrm{proj}_L(x) = M x$$

$$M = u u^T$$

Obs: $S \subseteq \mathbb{R}^n$. S subspace.

$u_1, \dots, u_l$ an orthonormal basis of S.

$$\mathrm{proj}_S(x) = (u_1 \cdot x)\, u_1 + \cdots + (u_l \cdot x)\, u_l$$

$$= \left(u_1 u_1^T + u_2 u_2^T + \cdots + u_l u_l^T\right) x$$

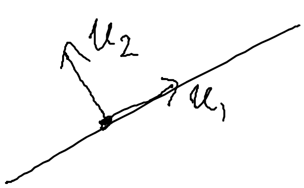
Claim $u_1 u_1^T + u_2 u_2^T + \cdots + u_l u_l^T = \begin{bmatrix} u_1 \cdots u_l \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \\ u_l^T \end{bmatrix}$

$Q = [u_1 \cdots u_l] \quad \mathrm{proj}_S(x) = Q Q^T x$

Let $u_1, \dots, u_l, u_{l+1}, \dots, u_n$ be an orthonormal basis of $\mathbb{R}^n$.

$1 \le k \le l$

$$\left(u_1 u_1^T + \cdots + u_l u_l^T\right) u_k = u_k$$



if $k \geq l+1$

$$\left(u_1 u_1^T + \cdots + u_l u_l^T\right) u_k = 0$$

if $k \le l$

$$\begin{bmatrix} & u \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \end{bmatrix} u_1 = [u_1 \quad u_l]\, e_1 = u_1$$

$$\begin{bmatrix} u_1 & \cdots & u_l \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \\ u_l^T \end{bmatrix} u_k = \begin{bmatrix} u_1 & \cdots & u_l \end{bmatrix} e_k = u_k$$

if $k \geq l+1$

$$\begin{bmatrix} u_1 & \cdots & u_l \end{bmatrix} \begin{bmatrix} u_1^T \\ \vdots \\ u_l^T \end{bmatrix} u_k = 0.$$

Ex: $S = \text{span}\left\{ \underbrace{\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}_{\text{orthonormal}} \right\}$

Let $Q = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$. Let $QQ^T = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$

$$QQ^T = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{proj}_S x = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Theorem: $A \in \mathbb{R}^{k \times n}$. Then

$$(\text{Im}(A))^\perp = \text{Ker}(A^T)$$

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

proof: $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$

$\mathrm{Im}(A) = \mathrm{Span}\{a_1, a_2, \dots, a_n\}$

$(\mathrm{Im}(A))^\perp = \{y \in \mathbb{R}^k : a_i^T y = 0 \text{ for all } i\}$

$\ker(A^T) = \{y \in \mathbb{R}^k : A^T y = 0\}$

$$A^T = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} \quad A^T y = \begin{bmatrix} a_1^T y \\ a_2^T y \\ \vdots \\ a_n^T y \end{bmatrix}$$

thus $(\mathrm{Im}(A))^\perp = \ker(A^T)$

<u>Example</u> $A = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$$\mathrm{Im}(A) = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}\right\} = \left\{\lambda \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} : \lambda \in \mathbb{R}\right\}$$

$$(\mathrm{Im}(A))^\perp = \text{solutions } \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$= \text{solutions to } x_1 - x_2 + x_3 = 0$$

$$\ker(A^T) = \ker(\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}) = \{x : x_1 - x_2 + x_3 = 0\}$$

<u>Theorem</u>. 1) $\ker(A) = \ker(A^T A)$

proof: if $x \in \ker(A) \Rightarrow Ax = 0 \Rightarrow$

proof: Let $x \in \ker(A) \Rightarrow Ax = 0 \Rightarrow$
$(A^T A)x = A^T(Ax) = A^T 0 = 0 \Rightarrow x \in \ker(A^T A)$
Thus $\ker(A) \subseteq \ker(A^T A)$.

Let $x \in \ker(A^T A) \Rightarrow A^T A x = 0$

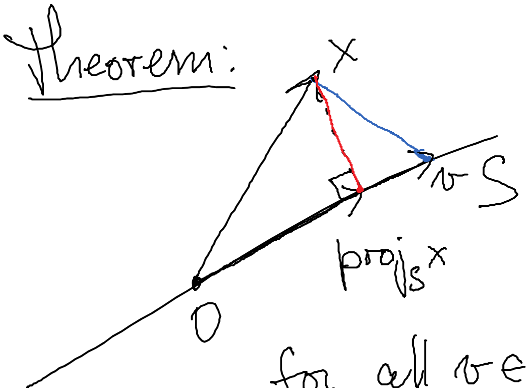
$$\|Ax\|^2 = (Ax)^T Ax = x^T \underbrace{A^T A x}_{=0} = x^T 0 = 0$$

Thus $Ax = 0$, thus $x \in \ker(A) \Rightarrow$
$\ker(A^T A) \subseteq \ker(A)$.
Then $\ker(A) = \ker(A^T A)$

2) $A \in \mathbb{R}^{k \times n}$ $\quad \underset{n\times k}{A^T}\ \underset{k \times n}{A} \in \mathbb{R}^{n \times n}$.

If $\ker(A) = 0 \Rightarrow A^T A$ is invertible.

Theorem:



$\text{proj}_S x$ is the vector in S closest to x

$$\|x - \text{proj}_S x\| < \|x - v\|$$

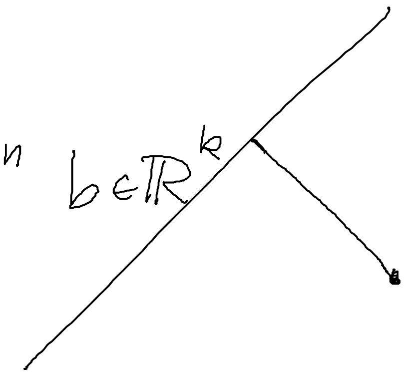
for all $v \in S$, with $v \neq \text{proj}_S x$.

proof $\quad x - v = \underbrace{x - \text{proj}_S x}_{\in S^\perp} + \underbrace{\text{proj}_S x - v}_{\in S}$

$\underbrace{\qquad}_{\in S^{\perp}} \quad \underbrace{\quad}_{\in S}$

then $\|x-v\|^2 = \|x-\text{proj}_S x\|^2 + \|\text{proj}_S x - v\|^2$

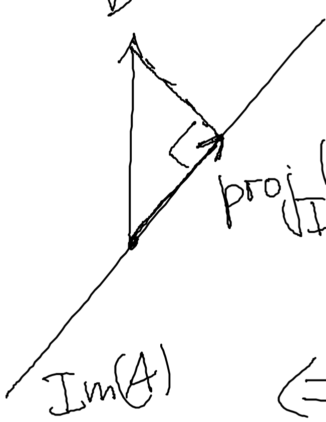
<u>Least squares</u>



$Ax = b \quad , \quad A \in \mathbb{R}^{k\times n} \quad b \in \mathbb{R}^k$

x^* is a least square solution of $Ax=b$
if $\|b-Ax^*\| \le \|b-Ax\|$ for all $x \in \mathbb{R}^n$

Ax^* is the vector in $\text{Im}(A)$ closest to b.



$\text{proj}_{\text{Im}(A)}(b) = Ax^*$

x^* is a least square solution of $Ax=b \iff$

$Ax^* = \text{proj}_{\text{Im}(A)}(b)$

$\iff b - Ax^* \in (\text{Im}(A))^{\perp}$

$\iff b - Ax^* \in \ker(A^T) \iff$

$A^T(b-Ax^*) = 0 \iff (A^TA)x^* = A^Tb$

<u>Theorem</u>: The least square solutions of $Ax=b$ are the solutions of $\boxed{(A^TA)\,x = A^Tb}$

These are called the normal equations.

Obs: If $Ker(A) = 0 \Rightarrow$ the least square solution of $Ax = b$ is $x^* = (A^TA)^{-1}A^T b$

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ $\quad b = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 6 & 6 \\ 6 & 14 & 18 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 2 & 6 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & -4 \\ 0 & 1 & 3 \end{array}\right] \quad x^* = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Obs: $A = [a_1 \ldots a_n]$ $\quad a_1, \ldots, a_n$ are l.i.

then $Ker(A) = \{0\} \Rightarrow x^* = (A^TA)^{-1}A^Tb$

$\Rightarrow proj_{Im(A)} b = A(A^TA)^{-1}A^T b$

If $v_1, \ldots, v_n$ is a basis of S, then the matrix of proj_S is $A(A^TA)^{-1}A^T$, where

$$A = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

Curve fitting

x	y
-1	0
0	1
1	2
2	-1

$$y = c_0 + c_1 x + c_2 x^2$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}}_{b}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 2 & 6 & 2 \\ 2 & 6 & 8 & 0 \\ 6 & 8 & 18 & -2 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 5 & 5 & -1 \\ 0 & 5 & 9 & -5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & -1/5 \\ \dots \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 1/2 & 0 & 2 \\ 0 & 1 & 0 & 4/5 \\ \dots \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1/2 & 1/2 & 1/2 \\ 0 & 1 & 1 & -1/5 \\ 0 & 0 & 4 & -4 \end{array}\right] \quad \left[\begin{array}{ccc|c} \ldots \\ 0 & 1 & 0 & 4/5 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 8/5 \\ 0 & 1 & 0 & 4/5 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad \begin{array}{l} c_0 = 8/5 \\ c_1 = 4/5 \\ c_2 = -1 \end{array}$$

$$y = -x^2 + \frac{4}{5}x + 8/5$$

<u>Example</u>

$$y = c_0 + c_1 x$$

x	y
x_1	y_1
$\vdots$	$\vdots$
x_n	y_n

$$\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}}_{A} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{b}$$

$$A^T A \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = A^T b$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & x_1 + \cdots + x_n \\ x_1 + \cdots + x_n & x_1^2 + \cdots + x_n^2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} y_1 + \cdots + y_n \\ y_1 x_1 + \cdots + y_n x_n \end{bmatrix}$$

$$\begin{bmatrix} x_1^2 + \cdots + x_n^2 & -(x_1 + \cdots + x_n) \end{bmatrix} \begin{bmatrix} y_1 + \cdots + y_n \end{bmatrix}$$

$\lfloor y_n \rfloor$

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \frac{1}{\left[n(x_1^2+\dots+x_n^2)-(x_1+\dots+x_n)^2\right]} \begin{bmatrix} x_1^2+\dots+x_n^2 & -(x_1+\dots+x_n) \\ -(x_1+\dots+x_n) & n \end{bmatrix} \begin{bmatrix} y_1+\dots+y_n \\ y_1x_1+\dots+y_nx_n \end{bmatrix}$$

Inner product spaces

or dot

<u>Def</u>: V a vector space. An inner product on V is a function $\langle\ ,\ \rangle$ defined for all pairs of elements of V such that:

1) $\langle f,f\rangle \geq 0$ and $=0 \iff f=0$
2) $\langle f,g\rangle = \langle g,f\rangle \quad \forall\, f,g\in V.$
3) $\langle \lambda f,g\rangle = \lambda\langle f,g\rangle$
4) $\langle f_1+f_2,g\rangle = \langle f_1,g\rangle + \langle f_2,g\rangle$

<u>Example</u> $V = C[0,1]$ = continuous real valued functions defined on $[0,1]$

$$\langle f,g\rangle = \int_0^1 f(x)g(x)\,dx$$