Ch 5

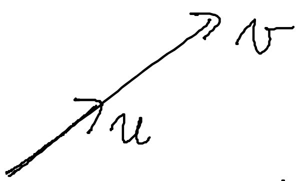
Monday, October 23, 2017 2:59 PM

Orthogonality

Def 1) $v, w \in \mathbb{R}^n$ are orthogonal (or perpendicular) if $v \cdot w = 0$.

2) the length or magnitude or norm of v is $\|v\| = \sqrt{v \cdot v}$

3) $u \in \mathbb{R}^n$ is called a unit vector if $\|u\| = 1$

Obs: 

$v \in \mathbb{R}^n$. u unit vector that points in the direction of v.

$u = \lambda v \quad \lambda > 0. \quad \|u\| = \underset{\lambda}{\underset{\shortparallel}{|\lambda|}} \|v\| = 1$

then $\lambda = \dfrac{1}{\|v\|}$

$$u = \frac{v}{\|v\|}$$

Def: $v_1, \dots, v_r \in \mathbb{R}^n$. They are orthonormal if:

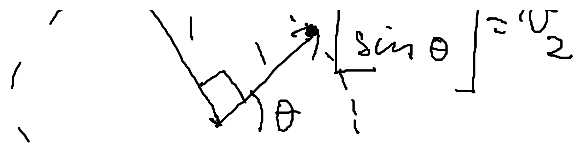
$$v_i \cdot v_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Ex 1) $e_1, e_2, \dots, e_n$ is an orthonormal set.

2) $v_1 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ $\quad v_1 \cdot v_2 = 0$

$\|v_1\| = \|v_2\| = 1$

$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = v_2$



Th: $v_1, \dots, v_r \in \mathbb{R}^n$ orthonormal. then:

1) they are linearly independent
2) If $r = n$ then $v_1, \dots, v_n$ is a basis of $\mathbb{R}^n$

proof: $0 = c_1 v_1 + \dots + c_r v_r$

dot product with v_k

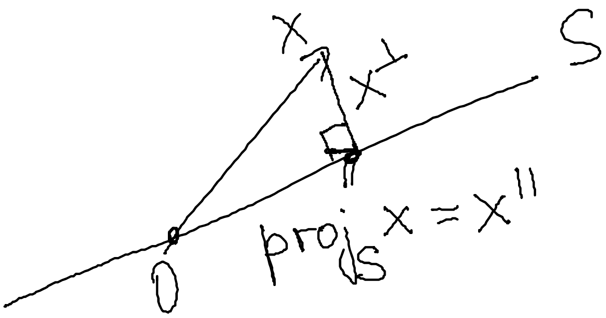
$$\underbrace{0 \cdot v_k}_{=0} = c_1\underbrace{(v_1 \cdot v_k)}_{=0} + c_2\underbrace{(v_2 \cdot v_k)}_{=0} + \dots \underbrace{\ }_{=c_k} \dots + c_n\underbrace{(v_n \cdot v_k)}_{=0}$$

$$\boxed{0 = c_k}$$

Orthogonal projections

$v_1, \dots, v_r$ basis of S.

$v_1, \dots, v_r$ orthonormal.



$$x = x^\parallel + x^\perp$$

$$x^\parallel = c_1 v_1 + \dots + c_r v_r$$

$$x = c_1 v_1 + \dots + c_r v_r + x^\perp \qquad x^\perp \cdot v = 0 \text{ for all } v \in S$$

Do dot product with v_k.

$$x \cdot v_k = c_1\underbrace{(v_1 \cdot v_k)}_{=0} + c_2\underbrace{(v_2 \cdot v_k)}_{=0} + \dots + c_k\overbrace{(v_k \cdot v_k)}^{=1} + \dots + c_n\overbrace{(v_n \cdot v_k)}^{=0} + \underbrace{x^\perp \cdot v_k}_{=0}$$

$$c_k = x \cdot v_k$$

$c_k = x \cdot v_k$

$$\boxed{\mathrm{proj}_S\, x = (v_1 \cdot x)\, v_1 + \cdots + (v_r \cdot x)\, v_r}$$

Before $S = \mathrm{Span}\{v\}$ $\quad \mathrm{proj}_S\, x = \dfrac{(v \cdot x)}{(v \cdot v)}\, v$

$$\boxed{x^\perp = x - \mathrm{proj}_S\, x}$$

Check that $x^\perp \cdot v_k = 0$

$$x^\perp \cdot v_k = x \cdot v_k - \underbrace{(\mathrm{proj}_S\, x) \cdot v_k}_{x \cdot v_k} = 0$$

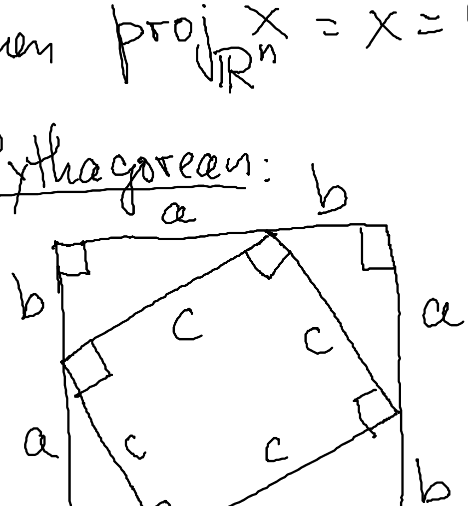
if $s \in S \Rightarrow s = \alpha_1 v_1 + \cdots + \alpha_r v_r$ because $v_1, \ldots, v_r$ is a basis of S. Thus,

r

$i=1$

Obs: If $v_1, \ldots, v_n$ is an orthonormal basis of $\mathbb{R}^n$, then $\mathrm{proj}_{\mathbb{R}^n}\, x = x = (v_1 \cdot x)\, v_1 + \cdots + (v_n \cdot x)\, v_n$

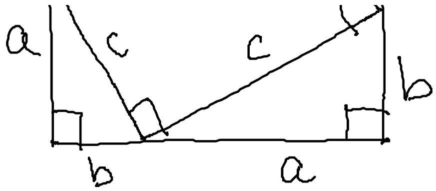
Pythagorean:



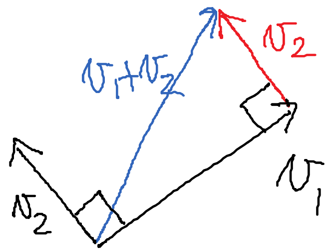
$$(a+b)^2 = c^2 + 4\,\frac{ab}{2}$$

big $\square$ = small $\square$ + 4 $\triangle$

$$a^2 + b^2 = c^2$$



$a^2+b^2=c^2$



$$\|v_1+v_2\|^2=(v_1+v_2)\cdot(v_1+v_2)=$$

$$v_1\cdot v_1+2\underbrace{v_1\cdot v_2}_{=0}+v_2\cdot v_2=\|v_1\|^2+\|v_2\|^2$$

Obs: $v_1,\dots,v_r$ orthogonal $\Rightarrow$

$$\|v_1+\dots+v_r\|^2=\|v_1\|^2+\dots+\|v_r\|^2$$

Obs: $v_1,\dots,v_r$ orthonormal basis of S subspace of $\mathbb{R}^n$. Let $x\in\mathbb{R}^n$.

$$\|x\|^2=\|x^\perp+\text{proj}_S x\|^2=\|x^\perp\|^2+\|\text{proj}_S x\|^2$$

$$\text{proj}_S x=(v_1\cdot x)v_1+\dots+(v_r\cdot x)v_r$$

$$\|\text{proj}_S x\|^2=\|(v_1\cdot x)v_1\|^2+\dots+\|(v_r\cdot x)v_r\|^2$$

$$\|\text{proj}_S x\|^2=(v_1\cdot x)^2+\dots+(v_r\cdot x)^2$$

Ex: $x=x_1e_1+\dots+x_ne_n$ $\qquad x\cdot e_j=x_j$

$$\|x\|^2=x_1^2+\dots+x_n^2$$

Obs: $\|\text{proj}_S x\| \le \|x\|$

Obs: $\text{proj}_S x : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation

$\mathbb{R}^n \to S$

proof: $x, y \in \mathbb{R}^n$

$$\text{proj}_S(x+y) = [v_1 \cdot (x+y)]v_1 + \dots + [v_r \cdot (x+y)]v_r$$

$$v_i \cdot (x+y) = v_i \cdot x + v_i \cdot y$$

$$[v_i \cdot (x+y)]\, v_i = (v_i \cdot x) v_i + (v_i \cdot y) v_i$$ add over all $1 \le i \le r$

$$\text{proj}_S(x+y) = \text{proj}_S x + \text{proj}_S y$$

$$\text{proj}_S(\lambda x) = \lambda\, \text{proj}_S x$$ (you can check) ✓

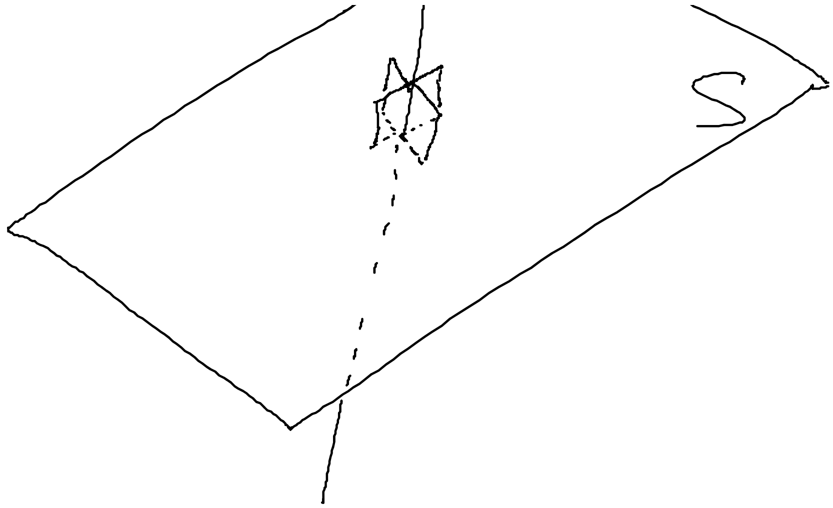
Obs: S subspace of $\mathbb{R}^n$.

$$\text{Im}(\text{proj}_S) = S$$

$$\text{Ker}(\text{proj}_S) = \{x \in \mathbb{R}^n : \text{proj}_S(x) = 0\} = S^\perp$$

$\text{proj}_S(x) = 0$ if $x = x^\perp$ that means $x \cdot s = 0$ for $s \in S$.

Def: $S^\perp = \{x \in \mathbb{R}^n : x \cdot s = 0 \text{ for all } s \in S\}$.



Def: $x \perp y$ if $x \cdot y = 0$

Obs: If $x \perp v_1, v_2, \dots, v_r \Rightarrow x \perp$ to any linear combination of $v_1, v_2, \dots, v_r \Rightarrow x \perp$ to any vector span$\{v_1, \dots, v_r\} \Rightarrow x \in (\text{Span}\{v_1, \dots, v_r\})^\perp$

because $x \cdot (c_1 v_1 + \dots + c_r v_r) = c_1 \underbrace{(x \cdot v_1)}_{=0} + \dots + c_r \underbrace{(x \cdot v_r)}_{=0}$

then $x \cdot (c_1 v_1 + \dots + c_r v_r) = 0$.

Notation $S^\perp$ = orthogonal complement of S.

Obs: If $v_1, \dots, v_r$ is a basis of S, then

$S^\perp = \{x : v_1 \cdot x = \dots = v_r \cdot x = 0\} =$

$= \text{Ker}\left(\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_r^T \end{bmatrix}\right) =$ in particular $S^\perp$ is a subspace

$= \mathrm{Ker}(\mathrm{proj}_S)$

<u>Obs</u>: 1) $S = \mathrm{Im}(\mathrm{proj}_S)$ $\quad S^\perp = \mathrm{Ker}(\mathrm{proj}_S)$

S subspace of $\mathbb{R}^n$ $\quad \mathrm{proj}_S : \mathbb{R}^n \to \mathbb{R}^n$

$$\dim(\mathrm{Im}(\mathrm{proj}_S)) + \dim(\mathrm{Ker}(\mathrm{proj}_S)) = n$$

$$\dim(S) + \dim(S^\perp) = n$$

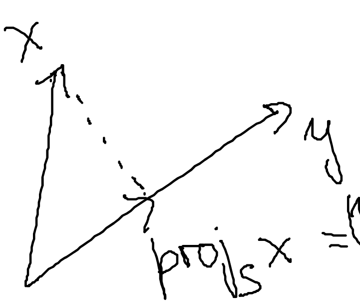
2) $S \cap S^\perp = \{0\}$

3) $(S^\perp)^\perp = S$

4) Any vector $x \in \mathbb{R}^n$ can be written uniquely in the form $x = s + s^\perp$ $\quad s \in S \quad s^\perp \in S^\perp$

$$s = \mathrm{proj}_S x \qquad s^\perp = x^\perp = x - \mathrm{proj}_S(x)$$

<u>Obs</u>: $\qquad S = \mathrm{span}\{y\}$



$$\mathrm{proj}_S x = \frac{(y \cdot x)\, y}{(y \cdot y)}$$

$$\|\mathrm{proj}_S x\| \le \|x\|$$

$$\frac{|y \cdot x|\, \|y\|}{(y \cdot y)} \le \|x\| \implies \frac{|x \cdot y|}{\|y\|} \le \|x\|$$

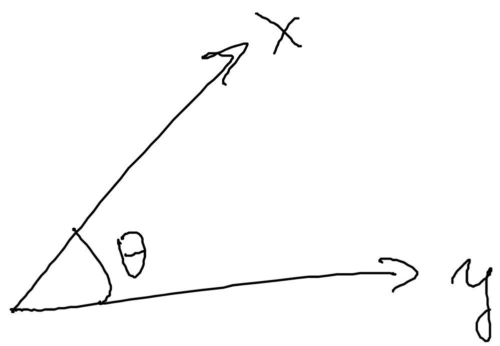
$(y \cdot y) = \|u\|^2$

$(y \cdot y) = \|y\|^2$ $\qquad\qquad$ $\|y\|$

Chaudy-Schwarz inequality

$|x \cdot y| \le \|x\| \, \|y\|$ (you have equality only when y is a multiple of x or one of the vectors is 0)

Obs:



$$\cos\theta = \frac{x \cdot y}{\|x\| \, \|y\|}$$

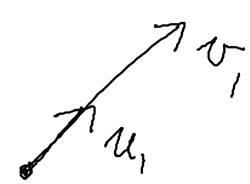
Ch 5.2 Gram-Schmidt

We are given $v_1, \dots, v_r$ linear independent vectors in $\mathbb{R}^n$.

Out put: $u_1, \dots, u_r$ orthonormal such that

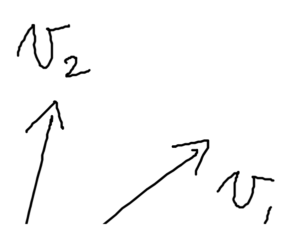
$\operatorname{Span}\{u_1, \dots, u_\ell\} = \operatorname{Span}\{v_1, \dots, v_\ell\}$ for all $1 \le \ell \le r$.

$\operatorname{Span}\{u_1\} = \operatorname{span}\{v_1\}$ then $\boxed{u_1 = \frac{v_1}{\|v_1\|}}$

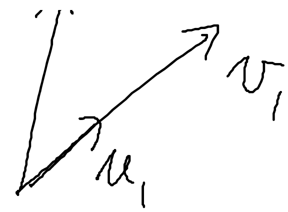


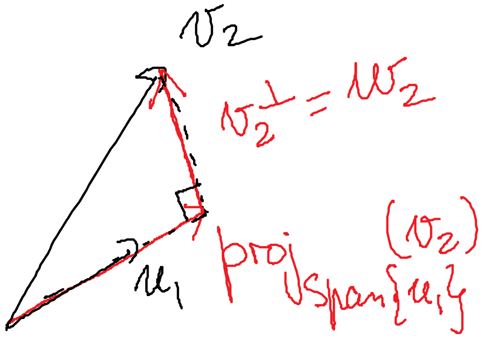
$\operatorname{Span}\{u_1\} = \operatorname{Span}\{v_1\}$ then

$\operatorname{Span}\{u_1, v_2\} = \operatorname{Span}\{v_1, v_2\}$



$\mathrm{Span}\{u_1, v_2\} = \mathrm{Span}\{v_1, v_2\}$





$u_2 = \dfrac{w_2}{\|w_2\|}$

$$\mathrm{proj}_{\mathrm{Span}\{u_1\}}(v_2) = (u_1 \cdot v_2)\, u_1$$

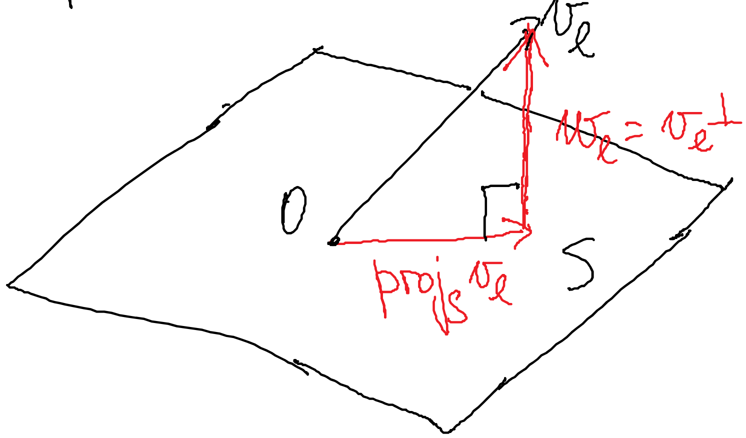
$$\boxed{w_2 = v_2 - (u_1 \cdot v_2)\, u_1}$$

$$\boxed{u_2 = \frac{w_2}{\|w_2\|}}$$

Once we have $u_1, \dots, u_{\ell-1}$, how do we get u_ℓ

$\mathrm{Span}\{v_1, \dots, v_{\ell-1}\} = \mathrm{Span}\{u_1, \dots, u_{\ell-1}\}$ add v_ℓ to get

$\mathrm{Span}\{v_1, \dots, v_\ell\} = \mathrm{Span}\{u_1, \dots, u_{\ell-1}, v_\ell\}$



$S = \mathrm{Span}\{u_1, \dots, u_{\ell-1}\}$

$u_\ell = \dfrac{w_\ell}{\|w_\ell\|}$

$$\mathrm{proj}_S v_\ell = (u_1 \cdot v_\ell) u_1 + \cdots + (u_{\ell-1} \cdot v_\ell) u_{\ell-1}$$

$$w_\ell = v_\ell - (u_1 \cdot v_\ell) u_1 - \cdots - (u_{\ell-1} \cdot v_\ell) u_{\ell-1}$$

$$u_\ell = \frac{w_\ell}{\|w_\ell\|}$$

Gram Schmidt

Input: $v_1, \ldots, v_r$ linearly idependent.

Output: $u_1, \ldots, u_r$ orthonormal.

$$u_1 = \frac{v_1}{\|v_1\|}$$

for $\ell = 2 : r$

$$w_\ell = v_\ell - (u_1 \cdot v_\ell) u_1 - \cdots - (u_{\ell-1} \cdot v_\ell) u_{\ell-1}$$

$$u_\ell = \frac{w_\ell}{\|w_\ell\|}$$

Property: $\mathrm{Span}\{u_1, \ldots, u_\ell\} = \mathrm{Span}\{v_1, \ldots, v_\ell\}$ for all $1 \le \ell \le r$.

<u>Example</u>: Apply Gram-Schmidt to $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}$

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad u_1 = \frac{v_1}{\|v_1\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \end{bmatrix} \qquad \|v_1\| \qquad \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad w_2 = v_2 - (u_1 \cdot v_2)\, u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$w_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - (1) \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \qquad u_2 = \frac{w_2}{\|w_2\|} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} \qquad w_3 = v_3 - (u_1 \cdot v_3)\, u_1 - (u_2 \cdot v_3)\, u_2$$

$$w_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right) \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/3 \end{bmatrix} - \left(\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} \right) \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$w_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + 2 \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} \qquad u_3 = \frac{w_3}{\|w_3\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

Result: $\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$

QR factorization

$k \geq n$

Input: $M \in \mathbb{R}^{k \times n}$ columns of M are linearly idependent

Output: $Q \in \mathbb{R}^{k \times n}$ and $R \in \mathbb{R}^{n \times n}$ such that

1) The columns of Q are orthonormal.

2) R is upper triangular $R = \begin{bmatrix} x & \cdots & x \\ & \ddots & \vdots \\ 0 & & x \end{bmatrix}$

3) $M = QR$

$$R = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix} \qquad r_1 = \begin{bmatrix} r_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad r_j = \begin{bmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ r_{jj} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

$$Q = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

$$v_j = Q r_j = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ r_{jj} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$v_j = r_{1j} u_1 + r_{2j} u_2 + \cdots + r_{jj} u_j$$

Then $\mathrm{Span}\{v_1, \ldots, v_\ell\} = \mathrm{Span}\{u_1, \ldots, u_\ell\}$ for all ℓ.

Before	Now	1a	1b

Before

$$u_1 = \frac{v_1}{\|v_1\|}$$

$$w_2 = v_2 - (u_1 \cdot v_2)\, u_1$$

$$\|w_2\|\, u_2 = v_2 - (u_1 \cdot v_2)\, u_1$$

$$v_2 = (u_1 \cdot v_2)\, u_1 + \|w_2\|\, u_2$$

Now

$$v_1 = r_{11} u_1$$

$$v_2 = r_{12} u_1 + r_{22} u_2$$

(1a) $r_{11} = \|v_1\|$ (1b) $u_1 = \frac{v_1}{r_{11}}$

$r_{12} = u_1 \cdot v_2$ (2a)

$w_2 = v_2 - r_{12} u_1$ (2b)

$r_{22} = \|w_2\|$ (2c)

$u_2 = \frac{w_2}{r_{22}}$ (2d)

$\vdots$

$$w_j = v_j - (u_1 \cdot v_j)\, u_1 - \cdots - (u_{j-1} \cdot v_j)\, u_{j-1}$$

$$r_{1j} = (u_1 \cdot v_j)$$

$$r_{2j} = (u_2 \cdot v_j)$$

$$\vdots$$

$$r_{j-1\, j} = (u_{j-1} \cdot v_j)$$

$$w_j = v_j - r_{1j} u_1 - \cdots - r_{j-1\, j}\, u_{j-1}$$

$$r_{jj} = \|w_j\|$$

$$u_j = \frac{w_j}{\|w_j\|}$$

Example $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ Compute the QR factorization

$$r_{11} = \|v_1\| = 2 \qquad u_1 = \frac{v_1}{r_{11}} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$\ldots$ $\begin{bmatrix} 1/2 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}$

$r_{11} = \|w_1\| = \quad\quad r_{11} = \quad \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$$r_{12} = u_1 \cdot v_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 1$$

$$w_2 = v_2 - r_{12} u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$r_{22} = \|w_2\| = 1$$

$$u_2 = \frac{w_2}{r_{22}} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$r_{13} = u_1 \cdot v_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} = 1$$

$$r_{23} = u_2 \cdot v_3 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} = -2$$

$$w_3 = v_3 - r_{13} u_1 - r_{23} u_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} + 2 \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$r_{33} = \|w_3\| = 1 \qquad u_3 = \frac{w_3}{r_{33}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$M = QR$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Def: $T: \mathbb{R}^n \to \mathbb{R}^k$ linear transformation.
We say that T is orthogonal if $\|T(x)\| = \|x\|$ for all $x \in \mathbb{R}^n$. If $T(x) = Ax$ is orthogonal, we say that A is orthogonal.

Obs: $T: \mathbb{R}^n \to \mathbb{R}^k$ an orthogonal linear transformation. If $x \cdot y = 0$ then $T(x) \cdot T(y) = 0$

proof: $\boxed{\|T(x+y)\|^2} = \|T(x) + T(y)\|^2 = (T(x) + T(y)) \cdot (T(x) + T(y)) = T(x) \cdot T(x) + 2\, T(x) \cdot T(y) + T(y) \cdot T(y) =$

$= \boxed{\|T(x)\|^2 + 2\,(T(x) \cdot T(y)) + \|T(y)\|^2}$

Since T is orthogonal, $\|T(x+y)\| = \|x+y\|$ and $\|T(x)\| = \|x\|$ and $\|T(y)\| = \|y\|$. Then

$$\|x+y\|^2 = \|x\|^2 + 2(T(x) \cdot T(y)) + \|y\|^2$$

Since $x \cdot y = 0$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

Since $x \cdot y = 0$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

then $T(x) \cdot T(y) = 0$.