$\mathbb{R}$ = set of real numbers

<u>Def</u>: A is a k by n matrix if A is of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \qquad a_{ij} \in \mathbb{R}$$

a_{ij} are the entries of A (or components).

$\mathbb{R}^{k \times n}$ = the set of all k by n matrices

<u>Example</u> $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 3}$

$a_{11} = 2 \quad a_{12} = 1 \quad a_{13} = -1$

$a_{21} = 3 \quad a_{22} = 1 \quad a_{23} = 0$

<u>Vectors</u>. An n-vector is an $n \times 1$ matrix

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad x_i \in \mathbb{R}$$

Sometimes we write vectors as rows

Sometimes we write ...

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (x_1, x_2, \dots, x_n) \qquad \mathbb{R}^n = \text{set of } n\text{-vectors}$$

Operations with matrices

Addition: $A \,\&\, B \in \mathbb{R}^{k \times n}$

$C = A + B \in \mathbb{R}^{k \times n}$ $\qquad c_{ij} = a_{ij} + b_{ij}$

Example
$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \end{bmatrix} + \begin{bmatrix} -2 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

Matrix-scalar multiplication

$\lambda \in \mathbb{R}$ $\quad A \in \mathbb{R}^{k \times n}$ then $B = \lambda A \in \mathbb{R}^{k \times n}$

$b_{ij} = \lambda a_{ij}$ $\quad$ **Example** $\quad 2\begin{bmatrix} -1 & 0 & 1 \\ -3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 \\ -6 & 4 & 2 \end{bmatrix}$

Properties $\quad A, B, C \in \mathbb{R}^{k \times n}$ $\qquad \lambda, \beta \in \mathbb{R}$

1) $A + B = B + A$
2) $(A + B) + C = A + (B + C)$
3) $(\lambda \beta) A = \lambda (\beta A)$

4) $\lambda(A+B) = \lambda A + \lambda B$

5) $(\lambda+\beta)A = \lambda A + \beta A$

Matrix-vector multiplication

Def: $A \in \mathbb{R}^{k\times n}$ $x \in \mathbb{R}^n$. Let a_j be the j^{th} column of A. $A = [a_1\ a_2 \ldots\ a_n]$ $a_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{kj} \end{bmatrix} \in \mathbb{R}^k$.

$$Ax = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

Note: $a_ix_i = x_ia_i$ for all i.

Example
$$\underbrace{\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}}_{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} 0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix}(-1) + \begin{bmatrix} 3 \\ -2 \end{bmatrix} 2 =$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 6 \\ -4 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

Obs: $A \in \mathbb{R}^{k\times n}$ $x \in \mathbb{R}^n$

$$Ax = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ \ldots + \cdots + a_{kn}x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix} \begin{bmatrix} \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \end{bmatrix}$$

<u>Example</u> $\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(0) + 0(-1) + 3(2) \\ -1(0) + 1(-1) + (-2)2 \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$

<u>Obs</u>: $A \in \mathbb{R}^{k \times n}$ $\quad x \in \mathbb{R}^n$ $\quad Ax \in \mathbb{R}^k$

<u>Matrix-matrix multiplication</u>

<u>Def</u>: $A \in \mathbb{R}^{k \times p}$ $\quad B \in \mathbb{R}^{p \times n}$ then $C = AB \in \mathbb{R}^{k \times n}$

$C = [c_1\, c_2 \cdots c_n]$ $\quad c_j$ is the j^{th} column of C

$B = [b_1\, b_2 \cdots b_n]$ $\quad b_j$ is the j^{th} column of B

$c_j = A b_j$

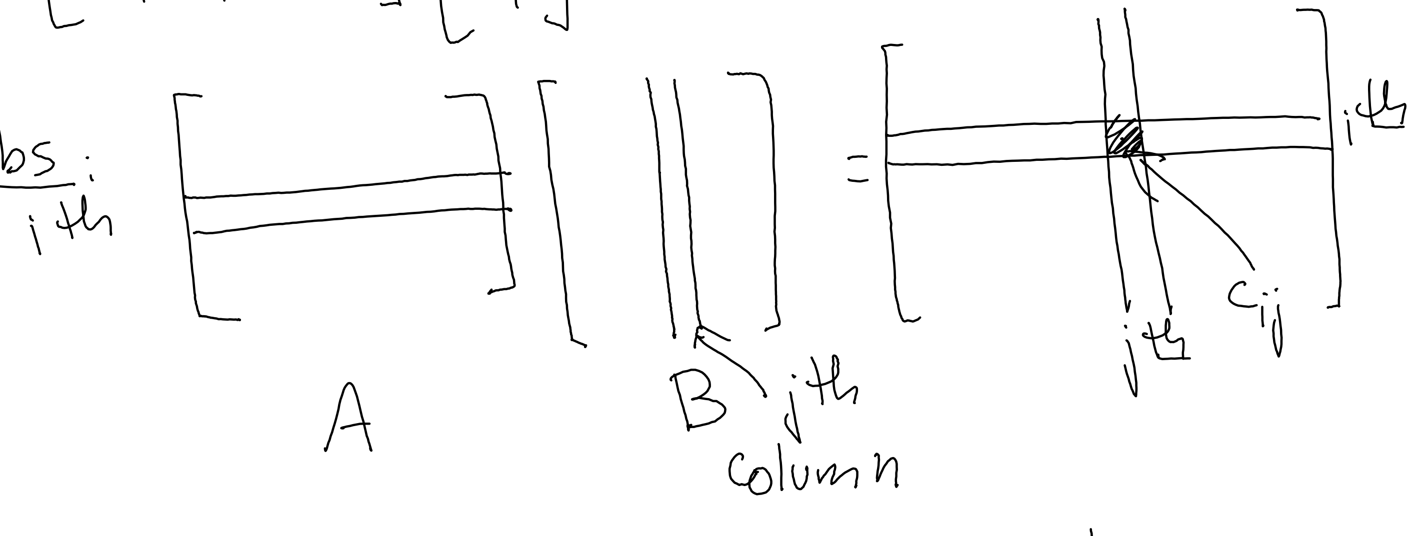
<u>Example</u> $\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 & 3 \\ & & \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

<u>Obs:</u>



$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots\cdots + a_{ip} b_{pj}$$

$$\begin{bmatrix} 2 & 0 & 3 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2(1)+0(0)+3(1) & 2(-1)+0(2)+3(1) \\ -1(1)+1(0)+(-2)1 & (-1)(-1)+1(2)+(-2)1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix}$$

<u>Properties:</u> 1) $A(B+C) = AB + AC$

2) $A(BC) = (AB)C$

3) $(A+B)C = AC + BC$

<u>Transpose of a matrix</u> $A \in \mathbb{R}^{k \times n}$. The transpose of

A^T is the i^{th}

Transpose of a matrix

A is $A^T \in \mathbb{R}^{n\times k}$. The j^{th} row of A^T is the j^{th} column of A.

Example: $\begin{bmatrix} 0 & 1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$

Properties 1) $(A^T)^T = A$

2) $(AB)^T = B^T A^T$

Notation: $[M]_{ij} = m_{ij}$ this is the (i,j) component of the matrix M. The number in the i^{th} row & j^{th} column of M.

Ex: $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$ $[A]_{21} = 1$

$A \in \mathbb{R}^{k\times p}$ $B \in \mathbb{R}^{p\times n}$

$[(AB)^T]_{ij} = [AB]_{ji} = a_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jp}b_{pi}$

$\ldots \,]_{ij} \quad \ldots_{ji} \quad + a_{jp} b_{pi}$

$$[B^T A^T]_{ij} = [B^T]_{i1} [A^T]_{1j} + [B^T]_{i2} [A^T]_{2j} + \cdots$$

$$+ [B^T]_{ip} [A^T]_{pj} = b_{1i} a_{j1} + b_{2i} a_{j2} + \cdots + b_{pi} a_{jp}$$

3) $(A+B)^T = A^T + B^T$

4) $(\lambda A)^T = \lambda A^T$

Special matrices

1) 0 is the matrix whose entries are all equal to 0.

2) Square matrix is a matrix with the same number of columns as the number of rows.

3) Triangular (these are square matrices)

$$\begin{bmatrix} x & \cdots & \cdots & x \\ 0 & x & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x \end{bmatrix} \qquad \begin{bmatrix} x & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ x & \cdots & \cdots & x \end{bmatrix}$$

upper $\qquad\qquad$ lower

Identity matrix

Upper

Diagonal $\begin{bmatrix} x & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & x \end{bmatrix}$ Identity matrix $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} = I$

Def: $A \in \mathbb{R}^{n \times n}$. A is symmetric if $A^T = A$

Ex: $\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ not symmetric $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ symmetric

Obs: $IA = A$ or $AI = A$

Def: A linear equation is an equation of the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are given numbers.

$x_1, x_2, \ldots, x_n$ are the unknowns or variables

Ex: $3x_1 + 2x_2 - x_3 = 1$

Def: A system of linear equations is of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad \vdots$$
$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = b_k$$

a_{ij} & b_j are known

<u>Goal</u>: Find all $x_1, x_2, \dots, x_n$ that satisfy the equations

In matrix vector notation

$A \in \mathbb{R}^{k\times n}$ $\quad b \in \mathbb{R}^k$ $\quad x \in \mathbb{R}^n$

A & b are given. Find all $x \in \mathbb{R}^n$ such that

$Ax = b$

<u>Def</u>: the augmented matrix of $Ax = b$ is

$$[A \,|\, b] \in \mathbb{R}^{k\times(n+1)}$$

<u>Example</u>: $2x_1 - x_3 = 1$

$x_1 - x_2 + 2x_3 = 2$

Augmented $\left[\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ 1 & -1 & 2 & 2 \end{array}\right]$

$x_1 - x_2 + \dots$

$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad$ Augmented matrix $\left[\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ 1 & -1 & 2 & 2 \end{array}\right]$

<u>Def</u>: A system is said to be consistent if it has at least one solution. Otherwise, it is inconsistent.

<u>Obs</u>: A systmem may have:

1) no solutions
2) only one solution
3) An infinite number of solutions.

<u>Def</u>: A matrix is said to be reduced row echelon (RRE) if it is of the form

$$\begin{bmatrix} 0 & \boxed{1} & x & 0 & x & x \\ 0 & 0 & 0 & \boxed{1} & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

1) the first non-zero entry in a row is equal to one. (in boxes) These are called leading ones. These entries are called pivots

the columns that contain a pivot entry is called a pivot column (In the example, columns 2 & 4 are the pivot columns. 1,2 & 2,4 are the pivot entries.

2) the pivot entry in a row is to the right of the pivot entry in the previous row.

3) If a row is zero, so is the next row.

4) the only non-zero entry is a pivot column is the leading one in that column

Solving $Ax=b$ when $[A|b]$ is RRE

Def: 1) x_i is a leading variable if the i^{th} column of $[A|b]$ is a pivot column.

2) x_i is a free variable if it is not a leading variable.

Step 1: label each free variable with a parameter, $t_1, t_2, \dots, t_r$

Step 2: Move to the right hand side all the terms with the free variables and replaced those variables by the corresponding parameters

The parameters can take any value, and this give us all the solutions.

Example

$$[A \mid b] = \left[\begin{array}{ccccc|} \boxed{1} & 2 & 0 & -1 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

$$x_1 + 2x_2 - x_4 = -2$$

$$x_1 = -2 - 2x_2 + x_4$$

$$x_1 = -2 - 2t_1 + t_2$$

$$x_1 = -2 - 2t_1 + t_2$$

$$x_2 = t_1$$

$$x_3 = 5 - t_2$$

$$x_4 = t_2$$

$$x_5 = 3 \qquad t_1, t_2 \in \mathbb{R}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 5 \\ 0 \\ 3 \end{bmatrix} + t_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

Def: Elementary operations

1) Multiply a row by a non-zero number.
2) Interchange two rows.
3) add to a row a multiple of an other row.

Obs: If we do an elementary operation on an

Obs: It we do an elementary operation on an augmented matrix we do not change the set of solutions

Gaussian elimination

Do elementary operations to change the augmented matrix into RRE form.

Examples 1) $2x_1 + 6x_2 + x_3 = 7$

$x_1 + 2x_2 - x_3 = -1$

$5x_1 + 7x_2 - 4x_3 = 9$

$$\left[\begin{array}{ccc|c} \boxed{2} & 6 & 1 & 7 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right]$$

$$\frac{R_1}{2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & 1/2 & 7/2 \\ 1 & 2 & -1 & -1 \\ 5 & 7 & -4 & 9 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - R_1 \\ R_3 - 5R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 1/2 & 7/2 \\ 0 & \boxed{-1} & -3/2 & -9/2 \\ 0 & -8 & -13/2 & -17/2 \end{array}\right]$$

$$\frac{R_2}{(-1)} \quad \left[\begin{array}{ccc|c} 1 & 3 & 1/2 & 7/2 \\ 0 & \boxed{1} & 3/2 & 9/2 \\ 0 & -8 & -13/2 & -17/2 \end{array}\right]$$

$$R_3 + 8R_2 \quad \left[\begin{array}{ccc|c} 1 & 3 & 1/2 & 7/2 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & \boxed{11/2} & 55/2 \end{array}\right]$$

$$\frac{2}{11}R_3 \quad \left[\begin{array}{ccc|c} 1 & 3 & 1/2 & 7/2 \\ 0 & 1 & 3/2 & 9/2 \\ 0 & 0 & \boxed{1} & 5 \end{array}\right]$$

$$\begin{array}{l} R_1 - \frac{1}{2}R_3 \\ R_2 - \frac{3}{2}R_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$R_1 - 3R_2 \quad \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 10 \\ 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & \boxed{1} & 5 \end{array}\right]$$

$x_1 = 10$

$x_2 = -3$

$x_3 = 5$